Exercise sheet 2
Theoretical Physics 3 : QM WS2020/2021
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Exercise 1. (60 points)

Consider a particle in an infinite square well potential of width l

$$V(x) = \begin{cases} 0 & \text{for} \quad -\frac{l}{2} \leq x \leq \frac{l}{2} \\ +\infty & \text{otherwise} \end{cases}$$

a) Starting from stationary Schrödinger equation, find the energy spectrum of the system.
Apart from the Schrödinger equation itself, which conditions determine the spectrum?
How does the ground state energy change when the width of the well is doubled?

b) The general solution of the time-dependent Schrödinger equation can be written in terms of superposition of stationary states $\Psi_n(x,t)$:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \Psi_n(x,t)$$

Write down the normalized set of (spatially) even and odd stationary states $\Psi_n(x,t)$.
Which additional conditions are required to fix a particular solution?
What freedom in determining the wave function is left after applying all physical conditions?

c) Consider time evolution of the following wave function:

$$\Psi(x, t=0) = \begin{cases} f(x) & \text{for} \quad 0 \leq x \leq \frac{l}{2} \\ 0 & \text{otherwise.} \end{cases}$$

In the most general case, independently of the exact form of $f(x)$, when will the probability to find the particle in the right half of the well ($0 \leq x \leq \frac{l}{2}$) become zero?
Write down the explicit expression of the wave function at these points of time.
In general, what is the total period of oscillation of the particle?

d) From now on (applies to tasks d), e) and f)), consider

$$f(x) = \begin{cases} \sqrt{\frac{96}{l^3}}\ x, & 0 \leq x \leq \frac{l}{4} \\ \sqrt{\frac{96}{l^3}}\ \left(\frac{l}{2} - x\right), & \frac{l}{4} \leq x \leq \frac{l}{2}. \end{cases}$$

Calculate expectation values of coordinate x and momentum p of the particle at $t = 0$.
How could one guess the results?

e) Calculate the product of standard deviations $\sigma_x \sigma_p$ at $t = 0$.
How does it compare to the case of the ground state of the system?

f) Assume we measure the energy of the particle in the given state.
What is the probability to find it in the ground state? (Recall that all Ψ_n are orthonormal.)
How does the probability to find the particle in the n-th energy state evolve with time? Which conclusion can then be drawn about the time evolution of the expectation value of the energy?

Exercise 2. (40 points)

A particle in the infinite square well has as its initial wave function an even mixture of the first and second excited stationary states:

$$\Psi(x,0) = A\,(\,\Psi_2(x) \,+\, \Psi_3(x))\ .$$

Note, that Ψ_2 is the spatially odd wave function of the first excited state of the system ($n=2$). While Ψ_3 is the spatially even wave function of the second excited state of the system ($n=3$).

a) Normalize $\Psi(x,0)$.

b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. To simplify the result, define $\omega \equiv \pi^2\hbar/2ml^2$.
Check that normalization holds with time.

c) Compute time evolution of $\langle x\rangle$.
What are the frequency and the amplitude of its oscillation?
Compute $\langle p\rangle$ using $\langle p\rangle = m\frac{\mathrm{d}}{\mathrm{d}x}\langle x\rangle$.

d) Find the expectation value of H. How does it compare to E_2 and E_3?