Exercise sheet 4
Theoretical Physics 3 : WS2020/2021
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Exercise 1. Fourier transform. (10 points)

Calculate the Fourier transforms of the following equations using the definition given in the lecture

a) *(2 p.)* $f(x) = \delta(x)$ and $f(x) = \delta(x - x_0)$

b) *(2 p.)* $f(x) = a = \text{const}$

c) *(3 p.)* $f(x) = \cos(x)$

d) *(3 p.)* $f(x) = \begin{cases} 1 - |t|, & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$

Exercise 2. Double δ-potential. (50 points)

Consider the following one-dimensional model potential for a molecule with one doubly degenerate state:

$$V(x) = -V_0 a \left(\delta(x - a) + \delta(x + a)\right),$$

where V_0 and a are real parameters.

a) *(5 p.)* Apply the Fourier transform to the corresponding Schrödinger equation, $\hat{H}(x)\psi(x) = E\psi(x)$. Show that in the momentum space it becomes

$$\frac{\hbar^2 k^2}{2m}\phi(k) - \frac{V_0 a}{\sqrt{2\pi}}\left(\psi(a)e^{-ika} + \psi(-a)e^{ika}\right) = E\phi(k),$$

where

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-ikx}\psi(x) \quad \text{and} \quad \psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}k\, e^{ikx}\phi(k).$$

Hint: Use $\int_{-\infty}^{\infty} \mathrm{d}x\, x e^{ax} = \frac{\partial}{\partial a}\int_{-\infty}^{\infty} \mathrm{d}x\, e^{ax}$

b) *(10 p.)* Using the obtained Schrödinger equation in the momentum space, find the bound states of the system in the coordinate space. How many bound states does the system have?
Hint: The solution must be consistent at the two points $x = \pm a$.

c) *(10 p.)* For $V_0 a = \frac{\hbar^2}{ma}$, find the energies of the stationary states and sketch the corresponding wave functions.
Hint: Use the fact that there are odd and even solutions.

d) *(10 p.)* Discuss the role of the parameter a on the stationary states (consider $a \to 0$ and $a \to \infty$).

e) *(15 p.)* Find the reflection and transmission coefficients for a beam of particles on this potential.

Exercise 3. Matrices. (40 points)

Recall that the matrix multiplication is a non-commutative operation. We define the commutator of two matrices A and B as $[A, B] = AB - BA$.

a) *(10 p.)* Prove the following identities for arbitrary matrices A, B and C:

$$[A, BC] = [A, B]C + B[A, C],$$
$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

b) *(10 p.)* Consider a specific case when $[A, [A, B]] = [B, [A, B]] = 0$.
Prove by induction, that

$$[A, B^n] = [A, B]nB^{n-1}.$$

c) *(10 p.)* We define a function of a matrix variable $f(A)$ through the Maclaurin series expansion (assuming it is possible):

$$f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} A^n.$$

In the case $[A, [A, B]] = [B, [A, B]] = 0$ show that

$$[A, F(B)] = [A, B]F'(B),$$

where F' is the function obtained by differentiation of F.

d) *(10 p.)* Consider the same case $[A, [A, B]] = [B, [A, B]] = 0$, and prove the Glauber formula

$$e^A \, e^B = e^{A+B} \, e^{\frac{1}{2}[A,B]}.$$

Hint: Consider the function $F(t) = e^{tA} \, e^{tB}$. Show that it has to satisfy the differential equation $\frac{\mathrm{d}F(t)}{\mathrm{d}t} = (A+B+t[A, B])F(t)$. Then solve the equation by noting that $(A+B)$ and $[A, B]$ commute, and, hence, can be treated as mere numbers.