Relativistic QFT (Theo 6a): Exercise Sheet 2
Total: 100 points

13/11/2020

1. Canonical quantization of the real Klein-Gordon field (20 points)

Using the following commutation relations for the creation-annihilation operators $\hat{a}_{\vec{p}}$, $\hat{a}^\dagger_{\vec{p}}$

$$[\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}] = [\hat{a}^\dagger_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}] = 0, \quad [\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}] = (2\pi)^3\delta^3(\vec{p}-\vec{q}), \tag{1}$$

prove the equal-time commutation relations

$$[\phi(t,\vec{x}), \phi(t,\vec{y})] = [\pi(t,\vec{x}), \pi(t,\vec{y})] = 0, \quad [\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \tag{2}$$

for the secondary quantized fields $\phi(x)$ and the momenta $\pi(x)$, which are

$$\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left[\hat{a}_{\vec{p}}e^{-ipx} + \hat{a}^\dagger_{\vec{p}}e^{+ipx}\right], \tag{3}$$

$$\dot{\phi}(t,\vec{x}) \equiv \pi(t,\vec{x}) = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left[\hat{a}_{\vec{p}}e^{-ipx} - \hat{a}^\dagger_{\vec{p}}e^{+ipx}\right] \tag{4}$$

2. Secondary quantized currents for the Klein-Gordon field (40 points)

(a) For the free real Klein-Gordon field $\phi(x)$, using (1) and (3) derive the secondary-quantized (expressed in terms of the secondary-quantized operators $a_{\vec{k}}$, $a^\dagger_{-\vec{k}}$):

- energy-momentum tensor $T^{\mu\nu}$,
- energy $E = \int d^4x\, T^{00}$,
- 3-momentum $\vec{P}^i = \int d^4x\, T^{0i}$,

Apply the normal ordering procedure. *For definitions of $T^{\mu\nu}$ and its derivation from symmetry w.r.t. spacetime translations see notes "Lecture 2" and "Noehter Current" on the website.* (**25 pts.**)

(b) For the two free real Klein-Gordon fields $\phi_i(x)$, $i = 1, 2$, with Lagrangian

$$\mathcal{L} = \sum_{i=1}^{2}\frac{1}{2}\left(\partial_\mu\phi_i\partial^\mu\phi_i - m^2\phi_i^2\right) \tag{5}$$

derive the secondary quantized expression for the charge that is related to the conserved current of rotational symmetry ($\phi_1 \to \quad \phi_1\cos\theta + \phi_2\sin\theta,\ \phi_2 \to -\phi_1\sin\theta + \phi_2\cos\theta$):

$$Q = \int dx(\phi_2\partial_t\phi_1 - \phi_1\partial_t\phi_2). \tag{6}$$

(**15 pts.**)

3. Ladder operators in Heisenberg and interaction picture (20 points)

Suppose we have some quantum Hamiltonian $\hat{H}$ that is defined by the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ of a quantum harmonic oscillator as

$$\hat{H} = \hat{H}_0 + \lambda(\hat{a}^\dagger + \hat{a}), \tag{7}$$

where $\hat{H}_0 = \omega[\hat{a}^\dagger \hat{a} + 1/2]$ is a quantum harmonic oscillator Hamiltonian, λ is a real parameter.

(a) Treating the term, proportional to λ as some interaction, obtain the ladder operators $\hat{a}$ and $\hat{a}^\dagger$ in the interaction picture, e.g. express

- $\hat{a}_I(t) = e^{it\hat{H}_0}\hat{a}e^{-it\hat{H}_0}$ in terms of $\hat{a}$,
- $\hat{a}_I^\dagger(t) = e^{it\hat{H}_0}\hat{a}^\dagger e^{-it\hat{H}_0}$ in terms of $\hat{a}^\dagger$

(b) Find the expressions for $\hat{a}$ and $\hat{a}^\dagger$ in the Heisenberg picture, e.g. express

- $\hat{a}_H(t) = e^{it\hat{H}}\hat{a}e^{-it\hat{H}}$ in terms of $\hat{a}$,
- $\hat{a}_H^\dagger(t) = e^{it\hat{H}}\hat{a}^\dagger e^{-it\hat{H}}$ in terms of $\hat{a}^\dagger$.

Hint: apply Feynman trick - differentiate the expression with respect to t.

4. Advanced and retarded propagators (20 points)

In principle there are four different possibilities to shift the poles in the complex p^0 energy plane for the Green function integral

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}. \tag{8}$$

Consider the following functions:

$$\Delta_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0 + i\epsilon)^2 - \vec{p}^2 - m^2} e^{-ip\cdot(x-y)}, \tag{9}$$

$$\Delta_A(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{(p^0 - i\epsilon)^2 - \vec{p}^2 - m^2} e^{-ip\cdot(x-y)}, \tag{10}$$

(a) Show that in comparison to the case of Feynman propagator,

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}, \tag{11}$$

when the positive energy pole is shifted below the real axis, and the negative energy pole is shifted above the real axis by the $i\epsilon$ prescription, both of the energy poles of Δ_R are shifted below the real axis, and both of the energy poles of Δ_A are shifted above the real axis. (**5 pts.**)

(b) Draw the contours of integration in cases when $x_0 - y_0 > 0$ and $x_0 - y_0 < 0$ for both of Δ_R and Δ_A. Show that Δ_A is nonzero only for positive time differences, and Δ_R - only for negative ones. (**15 pts.**)