Relativistic QFT (Theo 6a): Exercise Scheet 1
Total: 100 points

06/11/2020

1. Lorentz transformations (20 points)

Under active Lorentz transformation $\Lambda :\ x'^{\mu} = \Lambda^{\mu\nu} x_{\nu}$ fields transform as

$$\text{Scalar} : \Psi(x^{\mu}) \to \Psi([\Lambda^{-1}]^{\mu\nu} x_{\nu}), \qquad \text{Vector} : A^{\rho}(x^{\mu}) \to \Lambda^{\rho\sigma} A_{\sigma}([\Lambda^{-1}]^{\mu\nu} x_{\nu}) \tag{1}$$

Using the definition of a Lorentz transformation $\Lambda^{\mu\alpha} g_{\alpha\beta} \Lambda^{\nu\beta} = g^{\mu\nu}$, demonstrate that

(a) Klein-Grodon equation $(\partial_{\mu}\partial^{\mu} + m^2)\Psi = 0$ is invariant under a transformation Λ (**10 pts.**)

(b) Maxwell equations $\partial_{\mu} F^{\mu\nu} = 0$ with $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ are invariant under Λ (**10 pts.**)

2. Classical electrodynamics as a field theory (50 points)

It is convenient to consider classical electrodynamics in a Lorentz-invariant and gauge-invariant way, using the Lagrangian

$$\mathcal{L}_{EM} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{2}$$

where $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ is the electromagnetic (EM) tensor. It is antisymmetric, Lorentz and gauge invariant second-order tensor.

(a) The contravariant EM tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \equiv (\vec{E}, \vec{B}), \tag{3}$$

where $\vec{E}$ and $\vec{B}$ are vectors of electric and magnetic field respectively. Show that the covariant EM tensor $F_{\mu\nu} = (-\vec{E}, \vec{B})$.(**10 pts.**)

(b) Show that the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \left(\partial_{\mu} A^{\nu}\right) \left(\partial^{\mu} A_{\nu}\right) + \frac{1}{2} \left(\partial_{\mu} A^{\mu}\right)^2 \tag{4}$$

differs from $\mathcal{L}_{EM}$ by a total derivative.(**10 pts.**)

(c) Consider the Lagrangian for EM field with dual EM tensor

$$\mathcal{L} = -\frac{1}{2} F_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{5}$$

Find the equations of motion for the field A_μ. Show that they also give the Maxwell equations.(**15 pts.**)

(d) For $\mathcal{L}_{EM}$ obtain the canonical energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - g^{\mu\nu} \mathcal{L}. \tag{6}$$

Using the ambiguity in the energy-momentum definition as the Noether current

$$T^{\mu\nu} \to \tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho K^{\mu\nu\rho}, \quad K^{\mu\nu\rho} = -K^{\nu\mu\rho}, \tag{7}$$

symmetrize $T^{\mu\nu}$ choosing the appropriate $K^{\mu\nu\rho}$. (**15 pts.**)

3. Real scalar fields (30 points)

Consider the Lagrangian of two real scalar fields with equal masses:

$$\mathcal{L} = \sum_{i=1}^{2} \frac{1}{2} \left(\partial_\mu \phi_i \partial^\mu \phi_i - m^2 \phi_i^2 \right) \tag{8}$$

(a) Show that $\mathcal{L}$ is invariant under the 2D rotations, e.g. under the following field transformations

$$\phi_1 \to \phi_1 \cos\theta + \phi_2 \sin\theta \tag{9}$$

$$\phi_2 \to -\phi_1 \sin\theta + \phi_2 \cos\theta, \tag{10}$$

where $\theta \in [0, 2\pi]$ is the rotation angle.(**10 pts.**)

(b) Find the conserved current and charge of the theory, connected with the symmetry under rotations. (**10 pts.**)

(c) Rewrite the Lagrangian in terms of the one complex scalar field φ, applying the substitution

$$\varphi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}. \tag{11}$$

What symmetry does it have that corresponds to the rotation symmetry in case of the two real fields? Are there any changes in equations of motion?(**10 pts.**)