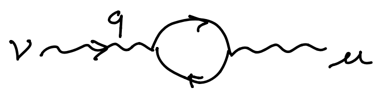
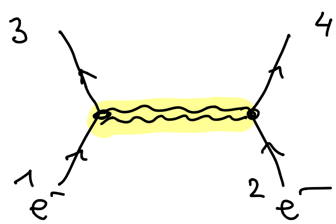
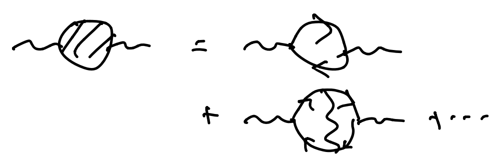
VI 2) RUNNING COUPLING IN QED

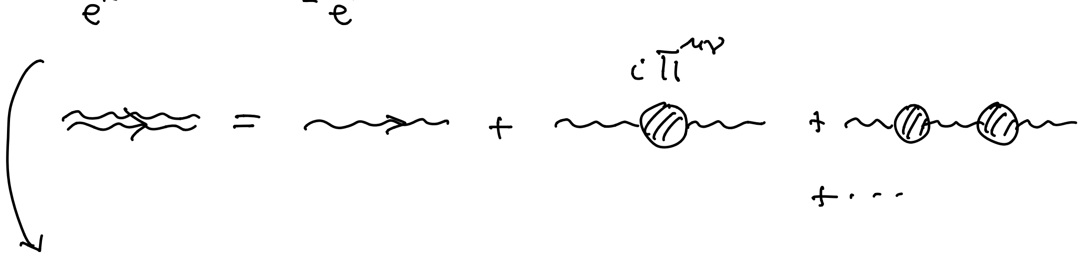


$i\Pi^{\mu\nu}(q)$

$$\boxed{\begin{aligned}\Pi^{\mu\nu}(q) &= \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)\\[1em] \Pi(q^2) &= -\frac{e^2}{12\pi^2}\Big(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \frac{1}{2}\,\rho'(\varsigma)\\ &\qquad + 6\int_0^1 dx\; x(1-x)\,\ln\frac{\mu^2}{m^2 - q^2 x(1-x)}\Big)\end{aligned}}$$







$$\mathcal{M} = i\,\bar{u}_3\gamma^\mu u_1 \cdot \bar{u}_4\gamma_\mu u_2 \quad \frac{e^2}{q^2\left[1 - \Pi(q^2)\right]}$$

$$V_c(q^2) = \frac{e^2}{q^2}$$

$$V(q^2) = \frac{e^2}{q^2\left[1 - \Pi(q^2)\right]}$$

$$\Pi(q^2) = \Pi(0) + \underbrace{\left(\Pi(q^2) - \Pi(0)\right)}_{q^2 \underbrace{\Pi'(0)}_{O(e^2)} + O(q^4)}$$

$$V(q^2) = \frac{e^2}{q^2\left[1 - \Pi(0) - \underbrace{\left(\Pi(q^2) - \Pi(0)\right)}\right]}$$

$$= \frac{e^2}{q^2\left[1 - \Pi(0)\right]} \left\{ 1 + \frac{\Pi(q^2) - \Pi(0)}{1 - \Pi(0)} + O(e^4) \right\}$$

$\uparrow$

POLE AT $q^2 = 0$ (PHOTON IS MASSLESS)

RESIDUE OF POLE $\dfrac{e^2}{1 - \Pi(0)} = e^2 Z_3$

Z_3 RENORMALIZATION CONSTANT

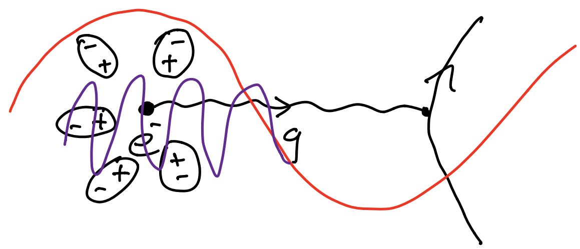
$$Z_3 = \frac{1}{1 - \Pi(0)} \simeq 1 + \Pi(0) + O(e^4)$$

$$= 1 - \frac{e^2}{12\pi^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \frac{1}{2} g'(4) + \ln \frac{\mu^2}{m^2} \right\} + O(e^4)$$

(ε FINITE)

$$V(q^2 \to 0) = \frac{e^2 Z_3}{q^2} \equiv \frac{e_R^2}{q^2} \quad \text{FINITE !}$$

RENORMALIZED CHARGE $\quad e_R^2 \equiv e^2 Z_3$



$$V_c \sim \frac{1}{r} e_R^2$$

↳ POTENTIAL AT FINITE q^2

$$V(q^2) = \frac{e_R^2}{q^2}\left\{ 1 + \underbrace{\frac{\Pi(q^2) - \Pi(0)}{1 - \Pi(0)}}_{Z_3 \frac{e^2}{2\pi^2} \int_0^1 dx\, x(1-x) \ln\left[1 - \frac{q^2}{m^2} x(1-x)\right]} + O(e^4) \right\}$$

$$\boxed{V(q^2) = \frac{e_R^2}{q^2}\left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx\; x(1-x) \ln\left[1 - \frac{q^2}{m^2} x(1-x)\right] + O(e_R^4) \right\}}$$

EFFECTIVE POTENTIAL

$$V(q^2) \equiv \frac{e_{eff}^2(q^2)}{q^2} \longleftarrow$$ EFFECTIVE CHARGE (q^2 DEP.)

$$e^2_{eff}(q^2) = e_R^2 \left\{ 1 + \frac{e_R^2}{2\pi^2} \int_0^1 dx \; x(1-x) \ln\left[1 - \frac{q^2}{m^2} x(1-x)\right] + O(e_R^4) \right\}$$

$$-q^2 = Q^2 > 0$$

$$Q^2 \nearrow \qquad e^2_{eff}(Q^2) \nearrow$$

$$Q^2 \gg m^2$$

$$e^2_{eff}(Q^2) \xrightarrow[Q^2 \gg]{} e_R^2 \left\{ 1 + \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2} + \dots \right\}$$

$$\boxed{e^2_{eff}(Q^2) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2}}}$$

$$e_R^2 \equiv e^2_{eff}(m^2)$$

$$\alpha_{em} \equiv \frac{e^2_{eff}}{4\pi}$$

$$\boxed{\alpha_{em}(Q^2) = \frac{\alpha_{em}(m^2)}{1 - \frac{1}{3\pi}\alpha_{em}(m^2) \ln \frac{Q^2}{m^2}}}$$

RUNNING COUPLING IN QED

$$\alpha_{em}(m^2) = \frac{1}{137}$$

Z-pole $\quad Q^2 \simeq (90\ \text{GeV})^2$

$$\frac{\alpha_{em}(m^2)}{3\pi}\ln\frac{Q^2}{m^2} \simeq 0.02$$

$$\alpha_{em}(90\ \text{GeV}^2) = \frac{1}{137}\cdot\left[1 + 0.02\right]$$

$\Rightarrow$ UEHLING POTENTIAL / LAMB SHIFT

$$\boxed{V(q^2) = \frac{e_R^2}{q^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left[1-\frac{q^2}{m^2}x(1-x)\right] + O(e_R^4)\right\}}$$

$-q^2 \ll m^2 \quad$ (LARGE DISTANCE LIMIT)

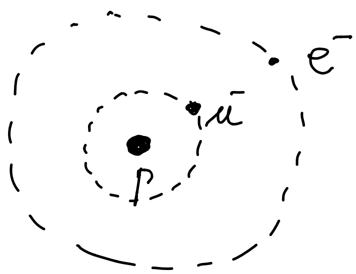
$$V(q^2) \underset{-q^2\ll m^2}{\simeq} \frac{e_R^2}{q^2}\left\{1 - \frac{e_R^2}{2\pi^2}\underbrace{\int_0^1 dx\; x^2(1-x)^2}_{1/30}\frac{q^2}{m^2} + \dots\right\}$$

$$= \frac{e_R^2}{q^2} - \frac{e_R^4}{60\pi^2 m^2} + O(e_R^6)$$

STATIC $\quad q^2 = -\bar{q}^2 \quad (q^0 = 0)$

$$V(r) = \int\frac{d^3\bar{q}}{(2\pi)^3}\, e^{i\bar{q}\cdot\bar{r}}\, V(\bar{q}^2)$$

$$= -\underbrace{\frac{e_R^2}{4\pi r}}_{\alpha_{em}} - \underbrace{\frac{e_R^4}{60\pi^2 m^2}\,\delta^3(\vec{r})}_{\text{UEHLING POT.}}$$



$$a_0 = \frac{1}{\alpha m_\ell}$$

$$\frac{m_\mu}{m_e} \simeq 200$$

BOHR

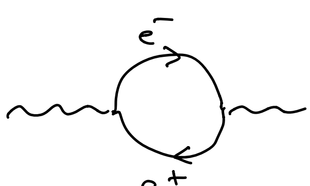
$$E_n = \int d^3\vec{r}\; \psi_n^* \left(T - \frac{\alpha_{em}}{r}\right) \psi_n = -\frac{1}{2} m\alpha^2 \frac{1}{n^2}$$

$$\Delta E = \int d^3\vec{r}\; \psi_n^*(\vec{r}) \left(-\frac{4}{15}\frac{\alpha_{em}^2}{m^2}\,\delta^3(\vec{r})\right) \psi_n(\vec{r})$$

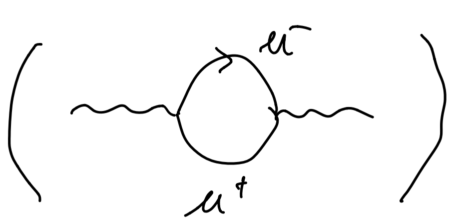
$$\Delta E_{nS} = -\frac{4}{15}\frac{\alpha_{em}^2}{m_e^2}\left|\psi_{nS}(0)\right|^2$$

ONLY NON-ZERO FOR $\ell = 0$

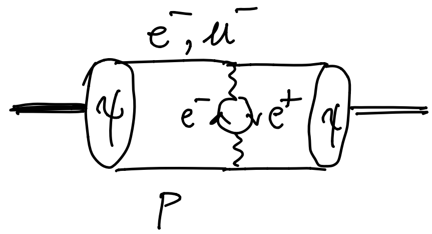
$$\left|\psi_{nS}(0)\right|^2 = \frac{(\alpha_{em} m_\ell)^3}{\pi n^3}$$



m_e

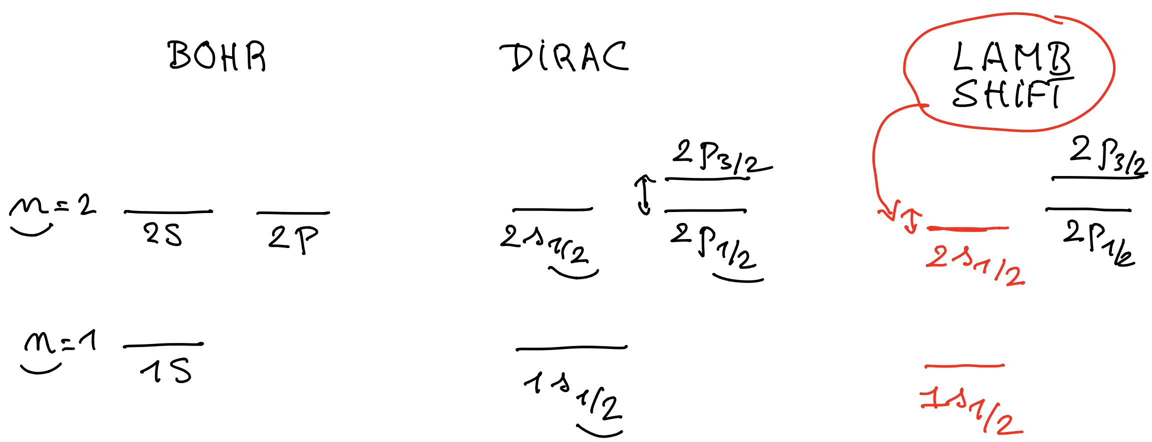


$$\frac{1}{m_e^2} \qquad \frac{1}{m_\mu^2}$$

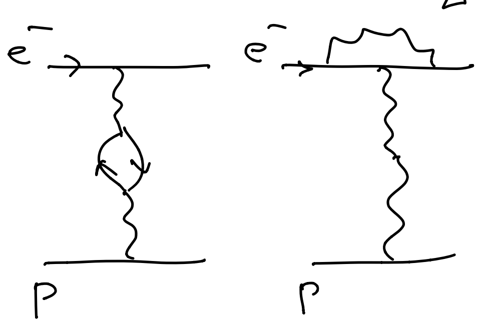


$$\boxed{\Delta E_{nS} = -\frac{4}{15}\,\frac{\alpha_{em}^5}{\pi n^3}\,\frac{m_\ell^3}{m_e^2}}$$

VACUUM POL. CONTR. TO LAMB SHIFT



H (e^-p) $m_\ell = m_e$ $\quad \Delta E_{2S} = -\frac{4}{15\pi}\,\frac{\alpha_{em}^5}{8}\,m_e$



$\sim -10^{-7}\,\mathrm{eV}$

$= h\nu$

$\nu \simeq 27\ \mathrm{MHz}$

(μ^-p) $m_\ell = m_\mu$ $\quad \Delta E_{2S} = -\frac{4}{15\pi}\,\frac{\alpha_{em}^5}{8}\,\frac{m_\mu^3}{m_e^2}$

$\sim -210\ \mathrm{meV}$