VI RENORMALIZATION OF QED

1) VACUUM POLARIZATION IN QED
2) RUNNING COUPLING IN QED
3) ANOMALOUS MAGNETIC MOMENT IN QED
4) ELECTRON SELF-ENERGY IN QED
5) RENORMALIZED PERTURBATION THEORY (QED
6) RENORMALIZABILITY : GENERAL

1) VACUUM POLARIZATION IN QED

⇒ SCALAR QED

↳ FEYNMAN RULES IN SCALAR QED

↳ SPIN-0 MATTER PARTICLES
e.g. π^+

$$\mathcal{L}_{KG} = (\partial_\mu \phi)^+ (\partial^\mu \phi) - m^2 \phi^+ \phi$$

BY REQUIRING THAT $\mathcal{L}$ IS INVARIANT UNDER $U(1)$ LOCAL PHASE TRANSFORMATION

$$\phi(x) \xrightarrow{U(1)} e^{i\chi(x)} \phi(x)$$

WE NEED TO REPLACE

$$\partial^\mu \to D^\mu = \partial^\mu + ieA^\mu$$

WITH

$$A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e}\partial^\mu \chi$$

$$\therefore \quad \mathcal{L} = (D_\mu \phi)^+ (D^\mu \phi) - m^2 \phi^+ \phi$$

$$= (\partial_\mu \phi^+ - ieA_\mu \phi^+)(\partial^\mu \phi + ieA^\mu \phi)$$

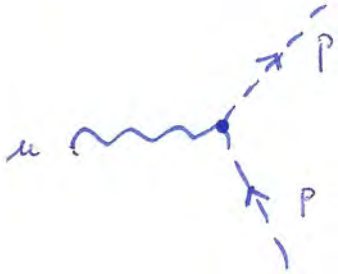
$$- m^2 \phi^+ \phi$$

$$= \mathcal{L}_{KG} + \mathcal{L}_{INT}$$

$$\mathcal{L}_{INT} = -ie\left(\phi^{+}(\partial^{\mu}\phi) - (\partial^{\mu}\phi^{+})\phi\right)A^{\mu} + e^{2} A_{\mu} A^{\mu} \phi^{+}\phi$$

2 TYPES OF INTERACTION TERMS FOR SCALAR QED

- LINEAR IN A^{μ}



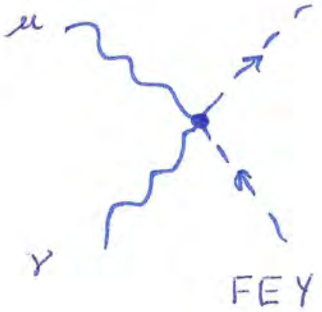
$$S = \mathbb{1} + \mathcal{M}\,(2\pi)^{4}\,\delta^{4}(\ldots)$$

$\mathcal{M}$: INVARIANT AMPLITUDE

$$\mathcal{M}_{fi} = e\,(-ip^{\mu} - ip'^{\mu})\,\varepsilon_{\mu}$$

FEYNMAN RULE: FOR VERTEX $\boxed{-ie\,(p+p')^{\mu}}$

- QUADRATIC IN A^{μ}



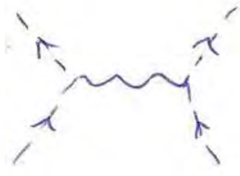
EACH A_{μ} FIELD CAN ABSORB INITIAL γ

$$\mathcal{M}_{fi} = +2ie^{2}\, g^{\mu\nu}\, \varepsilon_{\nu}\, \varepsilon^{*}_{\mu}$$

FEYNMAN RULE: FOR VERTEX $\boxed{+2ie^{2}\, g^{\mu\nu}}$

↳ 1-LOOP CORRECTION TO PHOTON PROPAGATOR

- LOWEST ORDER E.M. INTERACTION BETWEEN 2 CHARGED PARTICLE PROCEEDS THROUGH PHOTON PROPAGATOR

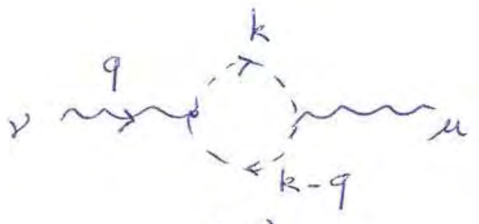


$$i D_0^{\mu\nu}(q) = \frac{-i g^{\mu\nu}}{q^2 + i\varepsilon}$$

(FEYNMAN GAUGE)

SUBSCRIPT ZERO DENOTES LOWEST ORDER

- 1° ORDER (IN e^2) CORRECTION TO PHOTON PROPAGATOR



(a)

+

(b)

$$\mathcal{M}_a^{\mu\nu} = \int \frac{d^4k}{(2\pi)^4} \frac{i.(-ie)(2k-q)^\nu \, i(-ie)(2k-q)^\mu}{(k^2 - m^2 + i\varepsilon)((k-q)^2 - m^2 + i\varepsilon)}$$

$$= e^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2k-q)^\nu (2k-q)^\mu}{(k^2 - m^2 + i\varepsilon)((k-q)^2 - m^2 + i\varepsilon)}$$

$$\mathcal{M}_b^{\mu\nu} = 2ie^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

$$= -2e^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon}$$

$$\mathcal{M}^{\mu\nu} = \mathcal{M}_a^{\mu\nu} + \mathcal{M}_b^{\mu\nu}$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 - m^2 + i\varepsilon\right]\left[(k-q)^2 - m^2 + i\varepsilon\right]}$$

$$\cdot \left\{ -4k^\mu k^\nu + 2k^\mu q^\nu + 2k^\nu q^\mu - q^\mu q^\nu + 2g^{\mu\nu}\left[(k-q)^2 - m^2\right] \right\}$$

- GAUGE INV. REQUIRES

$$q_\mu \mathcal{M}^{\mu\nu} = 0 \quad , \quad q_\nu \mathcal{M}^{\mu\nu} = 0$$

$$q_\mu \mathcal{M}^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[\cdots][\cdots]}$$

$$\cdot \left\{ -4(k.q)k^\nu + 2(k.q)q^\nu + 2q^2 k^\nu - q^2 q^\nu + 2q^\nu\left[(k-q)^2 - m^2\right] \right\}$$

- FEYNMAN PARAMETRIZATION

BRING INTEGRAL TO COMMON DENOMINATOR USING FEYNMAN PARAMETER TRICK

$$\boxed{\frac{1}{AB} = \int_0^1 dx \, \frac{1}{\left[A + (B-A)x\right]^2}}$$

PROOF. $\displaystyle\int_0^1 dx \, \frac{1}{\left[A + (B-A)x\right]^2}$

$$= \frac{-1}{(B-A)} \quad \frac{1}{A+(B-A)x} \Bigg|_0^1$$

$$= \frac{1}{(A-B)} \cdot \left\{ \frac{1}{B} - \frac{1}{A} \right\}$$

$$\overset{!}{=} \frac{1}{A.B}$$

- FOR $A = k^2 - m^2 + i\mathcal{E}$

$B = (k-q)^2 - m^2 + i\mathcal{E}$

$$\Downarrow$$

$$B - A = -2k.q + q^2$$

- $$\mathcal{M}^{\mu\nu} = -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 + (q^2 - 2k.q)x - m^2 + i\varepsilon\right]^2}$$

$$\cdot \left\{ -4k^\mu k^\nu + 2k^\mu q^\nu + 2k^\nu q^\mu - q^\mu q^\nu + 2g^{\mu\nu}\left[(k-q)^2 - m^2\right]\right\}$$

$$= -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[(k - qx)^2 + q^2 x(1-x) - m^2 + i\varepsilon\right]^2}$$

$$\cdot \left\{ \cdots \right\}$$

$$\Big\downarrow \qquad k' = k - qx$$

$$= -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 + q^2 x(1-x) - m^2 + i\varepsilon\right]^2}$$

$$\cdot \left\{ -4\left(k^\mu + q^\mu x\right)\left(k^\nu + q^\nu x\right) + 2\left(k^\mu + q^\mu x\right)q^\nu + 2\left(k^\nu + q^\nu x\right)q^\mu - q^\mu q^\nu + 2g^{\mu\nu}\left[\left(k - q(1-x)\right)^2 - m^2\right]\right\}$$

$$\Big\downarrow \qquad \int d^4k \frac{k^\alpha}{(k^2 + c)^2} = 0.$$

$$\mathcal{M}^{\mu\nu} = -e^2\int_0^1 dx \int\frac{d^4k}{(2\pi)^4}\,\frac{1}{\left[k^2+q^2x(1-x)-m^2+i\varepsilon\right]^2}$$

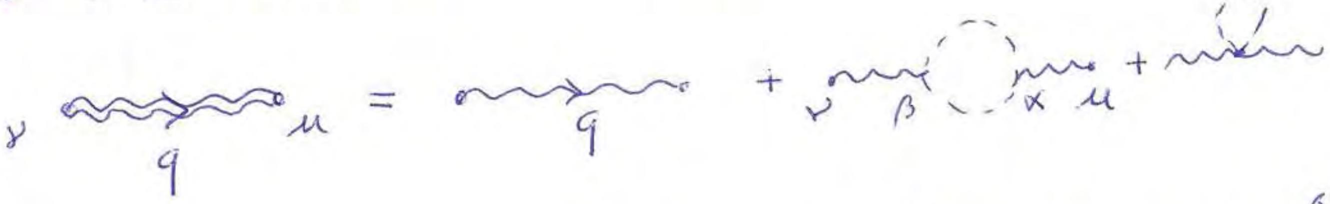
$$\cdot\Big\{-4k^\mu k^\nu$$

$$+\,q^\mu q^\nu\left[4x(1-x)-1\right]$$

$$+\,2g^{\mu\nu}\left[k^2+q^2(1-x)^2-m^2\right]\Big\}$$

<u>NOTE</u> $\int d^4k\,\frac{k^\mu k^\nu}{(k^2+c)^2}$ IS QUADRATICALLY DIVERGENT !

- FULL PHOTON PROPAGATOR



$$i\,D^{\mu\nu}(q) \equiv i\,D_0^{\mu\nu}(q) + i\,D_0^{\mu\alpha}(q)\,\mathcal{M}_{\alpha\beta}(q)\,i\,D_0^{\beta\nu}(q)$$

$$\underline{\underline{\mathcal{M}_{\alpha\beta}(q) \equiv i\,\Pi_{\alpha\beta}(q)}}$$

$$D^{\mu\nu}(q) = D_0^{\mu\nu}(q) - D_0^{\mu\alpha}(q)\,\Pi_{\alpha\beta}(q)\,D_0^{\beta\nu}(q)$$

$$\boxed{D^{\mu\nu}(q) = \frac{1}{q^2+i\varepsilon}\left\{-g^{\mu\nu} - \underset{\big\downarrow}{\Pi^{\mu\nu}(q)}\,\frac{1}{q^2}\right\}}$$

(FEYNMAN GAUGE)

VACUUM POLARIZATION CORRECTION

WITH

$$\boxed{\begin{aligned}\Pi^{\mu\nu}(q) = ie^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\,&\frac{1}{\left[k^2+q^2x(1-x)-m^2+i\varepsilon\right]^2}\\ &\cdot\left\{-4k^\mu k^\nu + q^\mu q^\nu\left[4x(1-x)-1\right]\right.\\ &\left.\quad + 2g^{\mu\nu}\left[k^2+q^2(1-x)^2-m^2\right]\right\}\end{aligned}}$$

BECAUSE k IS INTEGRATED OVER

$$\Pi^{\mu\nu}(q) = \Pi_1(q^2)\,q^2 g^{\mu\nu} + \Pi_2(q^2)\,q^\mu q^\nu$$

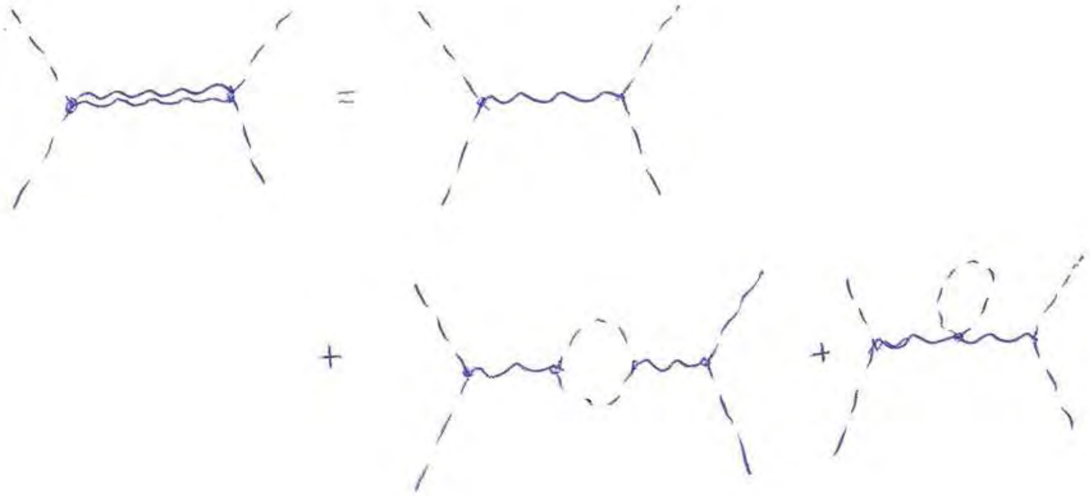
& GAUGE INVARIANCE $\quad q_\mu\Pi^{\mu\nu} = 0 = q_\nu\Pi^{\mu\nu}$

SCALAR FUNCTION

$$\boxed{\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2)}$$

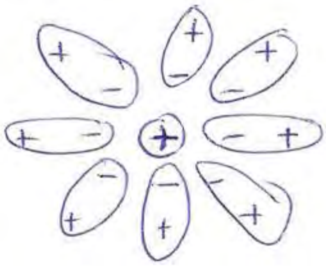
$$\Pi_1(q^2) \equiv \Pi(q^2)\ ,\quad \Pi_2(q^2) = -\Pi_1(q^2)$$

- DIVERGENCE OF VACUUM POL.



DUE TO VACUUM POL.

A TEST CHARGE IS POLARIZED IN VACUUM



TEST CHARGE → SURROUNDED BY DIPOLES
→ SCREENED : WE DON'T SEE THE "BARE" CHARGE
↳ IN ABSENCE OF VACUUM POL.

RENORMALIZATION: WE WILL ABSORB DIVERGENCE IN REDEFINING UNPHYSICAL "BARE" CHARGE

↳ REGULARIZATION

- CONSIDER INTEGRAL

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 - \Delta + i\varepsilon\right)^2}$$

FOR $k \to \infty$: I IS LOGARITHMICALLY DIVERGENT
AT HIGH-ENERGY
(UV DIVERGENCE)

IN ORDER TO CALCULATE LOOPS WITH $\mathcal{L}_{INT}$
WE NEED TO REGULARIZE THEORY
IN INTERMEDIATE STEP

- DIFFERENT REGULARIZATION METHODS :

1) CUT-OFF METHOD

$$\underset{\Downarrow}{\int d^4k} \longrightarrow \int^{\Lambda} d^4k$$

VIOLATES LORENTZ INVARIANCE

2) PAULI - VILLARS REGULARIZATION

$$\frac{1}{k^2 - m^2 + i\varepsilon} \longrightarrow \frac{1}{k^2 - m^2 + i\varepsilon} \cdot \underbrace{\left(\frac{m^2 - \Lambda^2}{k^2 - \Lambda^2 + i\varepsilon}\right)}$$

EXTRA FACTOR ← IMPROVES CONVERGENCE BY 2 POWERS

$\downarrow$ $\Lambda \to \infty$ AT END

$\underline{1}$

3) DIMENSIONAL REGULARIZATION. ('T HOOFT, VELTMAN)

CALCULATE LOOPS IN D - DIMENSIONS

$$\int d^4k \longrightarrow \int d^Dk$$

WITH $D < 4$ => IMPROVES CONVERGENCE

DIVERGENCE WILL APPEAR AS POLE

IN $\underline{\underline{\varepsilon \equiv 2 - \frac{D}{2}}}$ FOR $D < 4$, $\varepsilon > 0$

$\frac{1}{\varepsilon} \longrightarrow \infty$ IN LIMIT $D \to 4$

AT END : WHEN DIVERGENCE IS ABSORBED
BY REDEFINING BARE QUANTITY (CHARGE)
INTO PHYSICAL QUANTITY (OBSERVED)
ONE CAN LET $\varepsilon \to 0$ (ANALYTICAL CONTINUATION)

↳ EVALUATION OF INTEGRALS : WICK ROTATION

$$I = \int \frac{d^D k}{(2\pi)^D} \cdot \frac{1}{\left(k^2 - \Delta + i\varepsilon\right)^n}$$

- INTRODUCE SPHERICAL COORDINATES IN D DIM EUCLIDEAN SPACE

$$\theta_1 \ldots \theta_{D-2} : 0 \to \pi$$
$$\phi : 0 \to 2\pi$$

$$d^D k = |\bar{k}|^{D-1} \sin^{D-2}\theta_1 \cdots \sin\theta_{D-2}\, d|\bar{k}|\, d\theta_1 \ldots d\theta_{D-2}\, d\phi$$
$$= |\bar{k}|^{D-1} d|\bar{k}|\, d\Omega_D$$

- $d\Omega_D$: SOLID ANGLE ON UNIT SPHERE IN D-DIM EUCLIDEAN SPACE

$D=2$: $\displaystyle\int d\Omega_2 = \int_0^{2\pi} d\phi = 2\pi$ (CIRCLE)

$D=3$ $\displaystyle\int d\Omega_3 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta = 4\pi$ (SPHERE) ↓ 2 DIM SURFACE

$$D=4 \quad \int d\Omega_4 = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta_1 \sin^2\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2$$
$$= 4\pi \int_0^{\pi} d\theta_1 \sin^2\theta_1$$

$$\int d\Omega_4 = 4\pi \cdot \int_{-1}^{1} dx \; (1-x^2)^{1/2} \quad U = x^2$$

$$= 4\pi \int_0^1 dU \; U^{-1/2} (1-U)^{1/2}$$

USE BETA - FUNCTION

$$\boxed{B(n,m) = \int_0^1 dx \; x^{n-1} (1-x)^{m-1} = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}}$$

$$\int d\Omega_4 = 4\pi \; B\left(\frac{1}{2}, \frac{3}{2}\right) = 4\pi \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(2)}$$

$$\Gamma(1) = 1$$

$$\Gamma(n) = (n-1)! \quad \text{FOR INTEGER } n$$

$$\Gamma(x) = (x-1)\,\Gamma(x-1)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int d\Omega_4 = 4\pi \cdot \sqrt{\pi} \; \frac{1}{2} \sqrt{\pi} = 2\pi^2$$

(3 - SPHERE)

↓

3 DIM SURFACE

ARBITRARY D

$$\int_{(D-1)} d\Omega_D = 2\pi \int_0^{\pi} d\theta_1 \sin^{D-2}\theta_1 \cdots \int_0^{\pi} d\theta_{D-2} \sin\theta_{D-2}$$

$\uparrow$

D-1 DIM SURFACE

$$\text{WITH} \int_0^{\pi} d\theta \sin^n\theta = \int_{-1}^{1} dx \, (1-x^2)^{\frac{n-1}{2}}$$

$$= \int_0^1 du \; u^{-\frac{1}{2}} (1-u)^{\frac{n-1}{2}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = \sqrt{\pi} \, \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$\therefore \int_{(D-1)} d\Omega_D = 2\pi \, (\pi)^{\frac{D-2}{2}} \cdot \frac{\cancel{\Gamma\left(\frac{D-1}{2}\right)}}{\Gamma\left(\frac{D}{2}\right)} \cdot \frac{\cancel{\Gamma\left(\frac{D-2}{2}\right)}}{\cancel{\Gamma\left(\frac{D-1}{2}\right)}} \cdots \frac{\Gamma(1)}{\cancel{\Gamma\left(\frac{3}{2}\right)}}$$

$$\boxed{\int_{(D-1)} d\Omega_D = 2 \, \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}}$$

THROUGH ANALYTICAL CONTINUATION

RESULT ALSO FOR NON - INTEGER D

- WICK ROTATION

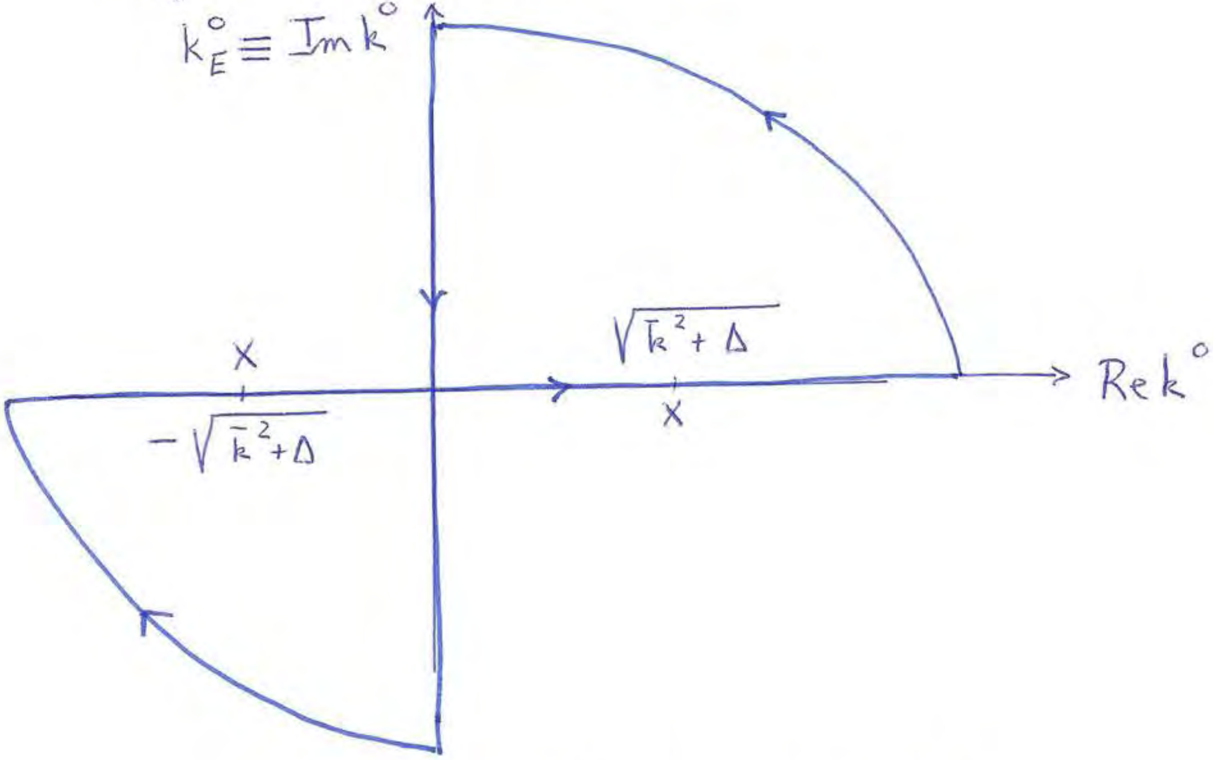
ASSUME $\Delta > 0$ (CAN ALSO BE SHOWN FOR $\Delta < 0$)

$$k^2 - \Delta + i\varepsilon = k_0^2 - \bar{k}^2 - \Delta + i\varepsilon$$
$$= \left(k_0 - \sqrt{\bar{k}^2+\Delta} + i\varepsilon\right)$$
$$\cdot \left(k_0 + \sqrt{\bar{k}^2+\Delta} - i\varepsilon\right)$$

INTEGRAL HAS POLES

AT $k_0 = \sqrt{\bar{k}^2+\Delta} - i\varepsilon$

$k_0 = -\sqrt{\bar{k}^2+\Delta} + i\varepsilon$



WE CAN REPLACE INTEGRAL ALONG $\mathrm{Re}\, k_0$ BY INTEGRAL ALONG $\mathrm{Im}\, k_0$: WICK ROTATION

(BECAUSE ALL POLES ARE OUTSIDE CONTOUR)

$$\underline{\underline{k^0 \equiv i k_E^0}}$$

k_E^0 : EUCLIDEAN ENERGY

$$k^\mu (k^0, \bar{k}) \qquad k^2 = (k^0)^2 - \bar{k}^2 = -k_E^2$$

$$k_E^\mu (k_E^0, \bar{k}) \qquad k_E^2 \equiv (k_E^0)^2 + \bar{k}^2$$

↳ EUCLIDEAN SPACE HAS UNIT METRIC

$$\int_{-\infty}^{+\infty} dk^0 \frac{1}{(k^2 - \Delta + i\varepsilon)^n} + i \int_{+\infty}^{-\infty} dk_E^0 \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n} = 0$$

$$\therefore \int_{-\infty}^{+\infty} dk^0 \frac{1}{(k^2 - \Delta + i\varepsilon)^n} = i \int_{-\infty}^{+\infty} dk_E^0 \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n}$$

$$I = i \int \frac{d^D k_E}{(2\pi)^D} \cdot \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n}$$

$\downarrow$ IN EUCLIDEAN SPACE ; WE CAN USE SPHERICAL COORDINATES IN D DIMENSIONS

$$I = i 2 \cdot \frac{\pi^{D/2}}{\Gamma(\frac{D}{2})} \cdot \frac{1}{(2\pi)^D} \int_0^\infty dk_E \frac{k_E^{D-1}}{(-k_E^2 - \Delta + i\varepsilon)^n}$$

$$\downarrow \quad \ell = k_E^2$$

$$I = \frac{i}{(4\pi)^{D/2}} \cdot \frac{(-1)^n}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty d\ell \, \frac{\ell^{\frac{D-2}{2}}}{(\ell + \Delta - i\varepsilon)^n}$$

$$\downarrow \qquad x \equiv \frac{\Delta}{\ell + \Delta} \qquad dx = -\frac{x^2}{\Delta}\, d\ell$$

$$1 - x = \frac{\ell}{\ell + \Delta}$$

NOTE Δ STANDS FOR $\Delta - i\varepsilon$

$$I = i\frac{(-1)^n}{(4\pi)^{D/2}} \cdot \frac{1}{\Gamma\left(\frac{D}{2}\right)} \int_0^1 dx \, \frac{\Delta}{x^2} (1-x)^{\frac{D}{2}-1} \left(\frac{x}{\Delta}\right)^{n - \frac{D}{2} + 1}$$

$$= \frac{i(-1)^n}{(4\pi)^{D/2}} \frac{1}{\Gamma\left(\frac{D}{2}\right)} \cdot \frac{1}{\Delta^{n-\frac{D}{2}}} \quad B\left(n - \frac{D}{2}, \frac{D}{2}\right)$$

$$= \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{\Delta^{n-D/2}} \cdot \frac{1}{\Gamma\left(\frac{D}{2}\right)} \cdot \frac{\Gamma\left(n - \frac{D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma(n)}$$

$$\boxed{I = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^n} = \frac{i}{(4\pi)^{D/2}} \cdot \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n - \frac{D}{2}}}}$$

- EXAMPLE $\underbrace{n = 2}$

$$I = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(k^2 - \Delta + i\varepsilon\right)^2} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \frac{1}{\Delta^{2-D/2}}$$

$$\downarrow \quad \varepsilon \equiv 2 - \frac{D}{2}$$

$$= \frac{i}{(4\pi)^2} \cdot \left(\frac{4\pi}{\Delta}\right)^{\varepsilon} \cdot \Gamma(\varepsilon)$$

USE: LAURENT SERIES EXPANSION OF Γ

$$\boxed{\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)}$$

EULER – MASCHERONI CONSTANT

$$\gamma_E \approx 0.577$$

$$\left(\frac{4\pi}{\Delta}\right)^{\varepsilon} = \exp\left\{\varepsilon \ln\left(\frac{4\pi}{\Delta}\right)\right\}$$
$$= 1 + \varepsilon \ln\left(\frac{4\pi}{\Delta}\right) + O(\varepsilon^2)$$

$$\therefore \quad \boxed{I = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} + \ln\left(\frac{4\pi}{\Delta}\right) - \gamma_E + O(\varepsilon) \right\}}$$

LOGARITHMIC DIVERGENCE OF I
IS EXPRESSED AS POLE AT $\varepsilon = 0$
$(D = 4)$

- FURTHER INTEGRALS

$$\hookrightarrow \boxed{\int \frac{d^D k}{(2\pi)^D} \cdot \frac{k^\mu}{(k^2 - \Delta + i\varepsilon)^n} = 0}$$

ODD FUNCTION INTEGRATED BETWEEN SYMM BOUNDARIES

$$\hookrightarrow \int \frac{d^D k}{(2\pi)^D} \cdot \frac{k^\mu k^\nu}{(k^2 - \Delta + i\varepsilon)^n} = A \, g^{\mu\nu}$$

ONLY RANK-2 SYMM. TENSOR POSSIBLE

$\downarrow$

MULTIPLY BY $g_{\mu\nu}$

NOTE : IN D-DIM : $g_{\mu\nu} g^{\mu\nu} = D$

$$A = \frac{1}{D} \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2 - \Delta)^n}$$

$$= \frac{1}{D} \int \frac{d^D k}{(2\pi)^D} \cdot \left\{ \frac{1}{(k^2 - \Delta)^{n-1}} + \frac{\Delta}{(k^2 - \Delta)^n} \right\}$$

$$= \frac{i}{(4\pi)^{D/2}} \frac{1}{D} \left\{ \frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n-1)} \frac{(-1)^{n-1}}{\Delta^{n-1-\frac{D}{2}}} + \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} \cdot \frac{(-1)^n \Delta}{\Delta^{n-\frac{D}{2}}} \right\}$$

$$= \frac{i}{(4\pi)^{D/2}} \frac{1}{D} \frac{(-1)^{n-1}}{\Delta^{n-1-\frac{D}{2}}} \cdot \frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n)} \underbrace{\left\{ n-1-(n-1-\frac{D}{2}) \right\}}_{D/2}$$

$$\boxed{\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - \Delta + i\varepsilon)^n} = \frac{i g^{\mu\nu}}{(4\pi)^{D/2}} \cdot \frac{\Gamma(n-1-\frac{D}{2})}{2\Gamma(n)} \cdot \frac{(-1)^{n-1}}{\Delta^{n-1-D/2}}}$$

NOTE

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left\{ \frac{1}{\varepsilon} + \left(1 + \dots + \frac{1}{n}\right) - \gamma_E + O(\varepsilon) \right\}$$

$n = 1, 2, 3, \dots$

↳ FIELD DIMENSIONS IN D-DIM SPACE

WHEN GOING TO D-DIM SPACE WE CHANGE DIMENSIONS OF FIELDS

$$\mathcal{L} = (\partial_\mu \phi)^+ (\partial^\mu \phi) - m^2 \phi^+ \phi$$
$$- ie\left(\phi^+(\partial^\mu \phi) - (\partial^\mu \phi)^+ \phi\right) A_\mu - e^2 A_\mu A^\mu \phi^+ \phi$$

- D = 4

$$S^{(1)} = i \int d^4x \, \mathcal{L}^{(1)}_{INT}$$

MASS DIMENSION [ ]

$[S] = 0$ S-MATRIX : DIMENSIONLESS

$[\mathcal{L}] = 4$

$[\phi] = 1$

$[A^\mu] = 1$

$[e] = 0$

- D ≠ 4

$[S] = 0$

$[d^Dx] = -D$

$[\mathcal{L}] = D$

$[\phi] = \frac{D-2}{2}$

$[A^\mu] = \frac{D-2}{2}$

$[e] = 2 - \frac{D}{2} = \varepsilon$

TO AVOID DIMENSIONAL COUPLING WHEN $D \neq 4$

INTRODUCE AN ARBITRARY MASS SCALE μ

(RENORMALIZATION SCALE)

$$\underline{\underline{e \xrightarrow{\text{IN D-DIM}} \mu^{\varepsilon} e}}$$

AN TREAT e FURTHER AS DIMENSIONLESS.

$$\int \frac{d^4k}{(2\pi)^4} \frac{e^2}{(k^2 - \Delta + i\varepsilon)^2} \xrightarrow[\text{REG}]{\text{DIM.}} \mu^{2\varepsilon} e^2 \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - \Delta^2 + i\varepsilon)^2}$$

NOTE : EVEN THOUGH $\mu^{2\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 1$

WE NEED TO EXPAND FULL INTEGRAL AROUND $\varepsilon = 0$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - \Delta^2 + i\varepsilon)^2} = \frac{a}{\varepsilon} + b + O(\varepsilon)$$

$$\mu^{2\varepsilon} = 1 + \varepsilon \ln \mu^2 + O(\varepsilon^2)$$

$$\mu^{2\varepsilon} \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - \Delta^2 + i\varepsilon)^2} = \frac{a}{\varepsilon} + \underset{\substack{\uparrow \\ \text{GIVES A} \\ \text{FINITE} \\ \text{CONTRIBUTION}}}{\left(b + \ln \mu^2\right)} + O(\varepsilon)$$

↳ EVALUATION OF VACUUM POLARIZATION 1-LOOP INT.

FROM PAGE 8

- $$\Pi^{\mu\nu} = i e^2 \mu^{2\varepsilon} \int_0^1 dx$$
$$\cdot \left\{ -4 \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2-\Delta)^2} \right.$$
$$+ 2 g^{\mu\nu} \int \frac{d^D k}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^2}$$
$$+ \left(q^\mu q^\nu \left[4x(1-x) - 1 \right] + 2 g^{\mu\nu} \left[q^2 (1-x)^2 - m^2 \right] \right)$$
$$\left. \cdot \int \frac{d^D k}{(2\pi)^D} \cdot \frac{1}{(k^2-\Delta)^2} \right\}$$

WITH $\underline{\underline{\Delta \equiv m^2 - q^2 x(1-x)}}$

- $$\Pi^{\mu\nu} = i e^2 \mu^{2\varepsilon} \int_0^1 dx \cdot \frac{i}{(4\pi)^{D/2}}$$
$$\cdot \left\{ + 4 g^{\mu\nu} \frac{\Gamma(-1+\varepsilon)}{2} \cdot \frac{1}{\Delta^{-1+\varepsilon}} \right.$$
$$- 2 g^{\mu\nu} D \frac{\Gamma(-1+\varepsilon)}{2} \frac{1}{\Delta^{-1+\varepsilon}}$$
$$\left. + \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon} \left(q^\mu q^\nu \left[4x(1-x) - 1 \right] + 2 g^{\mu\nu} \left[q^2 (1-x)^2 - m^2 \right] \right) \right\}$$

$$\Pi^{\mu\nu} = -\frac{e^2\mu^{2\varepsilon}}{(4\pi)^{D/2}}\int_0^1 dx\,\cdot$$

$$\cdot\left\{ 2g^{\mu\nu}(-1+\varepsilon)\frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}} + 2g^{\mu\nu}\frac{\Gamma(\varepsilon)}{\Delta^{\varepsilon}}\left[q^2(1-x)^2 - m^2\right] + q^\mu q^\nu \frac{\Gamma(\varepsilon)}{\Delta^{\varepsilon}}\left[4x(1-x)-1\right]\right\}$$

$$\Big\downarrow \qquad (-1+\varepsilon)\,\Gamma(-1+\varepsilon) = \Gamma(\varepsilon)$$

$$\Pi^{\mu\nu} = -\frac{e^2\mu^{2\varepsilon}}{(4\pi)^2}\,\Gamma(\varepsilon)\,(4\pi)^{\varepsilon}$$

$$\cdot\int_0^1 dx\,\frac{1}{\left[m^2-q^2x(1-x)\right]^{\varepsilon}}\left\{2g^{\mu\nu}\underbrace{\left[\cancel{m^2}-q^2x(1-x) + q^2(1-x)^2-\cancel{m^2}\right]}_{q^2(1-x)(1-2x)} + q^\mu q^\nu\left[-2(1-x)(1-2x) + \cancel{(1-2x)}\right]\right\}$$

$\Big\downarrow$ NOTE: DUE TO SYMMETRY OF INTEGRAL (AROUND $x=\frac{1}{2}$)

$$\int_0^1 dx\,\frac{(1-2x)}{\left[m^2-q^2x(1-x)\right]^{\varepsilon}} = 0$$

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right)$$

$$\cdot \frac{(-e^2)\mu^{2\varepsilon}}{(4\pi)^2} \, 2 \, \Gamma(\varepsilon) \, (4\pi)^\varepsilon \int_0^1 dx \, \frac{(\cancel{1} - x)(1-2x)}{\left[m^2 - q^2 x (1-x)\right]^\varepsilon}$$

NOTE : GAUGE INV. FACTOR $\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$
COMES OUT EXPLICITELY

- EXPANSION IN ε AROUND $\varepsilon = 0$

$$\Pi^{\mu\nu}(q) = - \frac{e^2}{8\pi^2} \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right)$$

$$\cdot \left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) + O(\varepsilon) \right) \left(1 + \varepsilon \ln \mu^2 \right)$$

$$\cdot \int_0^1 dx \, x \, (2x-1) \left(1 - \varepsilon \ln \left[m^2 - q^2 x (1-x) \right] \right)$$

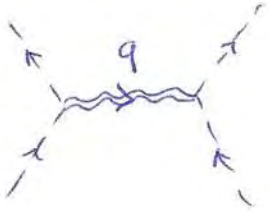
$$\therefore \quad \boxed{\begin{aligned} \Pi^{\mu\nu}(q) = {} & - \frac{e^2}{8\pi^2} \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \\ & \cdot \int_0^1 dx \, x \, (2x-1) \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi \mu^2}{m^2 - q^2 x(1-x)} \right) \right. \\ & \qquad \left. + \, O(\varepsilon) \right\} \end{aligned}}$$

$$\boxed{\begin{aligned}\Pi^{\mu\nu}(q) &= -\frac{e^2}{48\pi^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\\ &\cdot\left\{\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \int_0^1 dx\; x(2x-1)\ln\left(\frac{\mu^2}{m^2 - q^2 x(1-x)}\right)\right\}\end{aligned}}$$

$\downarrow$

FOR SPACELIKE PHOTON $(q^2 = -Q^2 < 0)$ $\quad Q^2 > 0$

e.g. iN



AND ASSUME $Q^2 \gg m^2$

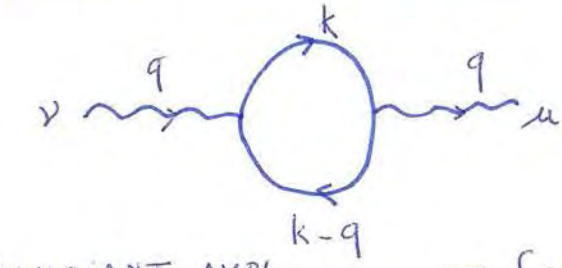
$$\int_0^1 dx\; x(2x-1)\ln\frac{\mu^2}{Q^2 x(1-x)}$$

$$= \frac{1}{6}\ln\frac{\mu^2}{Q^2} - \underbrace{\int_0^1 dx\; x(2x-1)\ln x(1-x)}_{-\frac{4}{9}}$$

$$\boxed{\begin{aligned}\Pi^{\mu\nu}(q) &\overset{Q^2\gg m^2}{\simeq} -\frac{e^2}{48\pi^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\\ &\cdot\left\{\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{Q^2} + \frac{8}{3}\right\}\end{aligned}}$$

⇒ SPINOR QED

↳ VACUUM POLARIZATION



INVARIANT AMPL.

$$i\Pi^{\mu\nu}(q) = (-1)\int \frac{d^4k}{(2\pi)^4} \frac{Tr\left\{(ie\gamma^\mu)\, i(\not k + m)\,(ie\gamma^\nu)\, i(\not k - \not q + m)\right\}}{\left[k^2 - m^2 + i\varepsilon\right]\left[(k-q)^2 - m^2 + i\varepsilon\right]}$$

CLOSED FERMION LOOP

$$\Pi^{\mu\nu}(q) = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left\{\gamma^\mu(\not k + m)\,\gamma^\nu(\not k - \not q + m)\right\}}{\left[k^2 - m^2 + i\varepsilon\right]\left[(k-q)^2 - m^2 + i\varepsilon\right]}$$

$\downarrow$ FEYNMAN PARAMETER & $k \to k' = k - qx$

$$= ie^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left\{\gamma^\mu(\not k + \not q x + m)\,\gamma^\nu(\not k - \not q(1-x) + m)\right\}}{\left[k^2 - \left(m^2 - q^2 x(1-x)\right) + i\varepsilon\right]^2}$$

$$\downarrow \quad \int \frac{d^4k}{(2\pi)^4} \frac{\not k}{\left[k^2 - \Delta + i\varepsilon\right]^2} = 0$$

TERMS LINEAR IN k IN NUMERATOR GIVE 0 UPON INTEGRATION

$$\boxed{\begin{aligned}\Pi^{\mu\nu}(q) = i e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \, & \frac{1}{\left[k^2 - \left(m^2 - q^2 x(1-x)\right) + i\varepsilon\right]^2} \\ & \cdot \Big\{ \operatorname{Tr}\left\{\gamma^\mu \not{k}\, \gamma^\nu \not{k}\right\} \\ & - x(1-x) \operatorname{Tr}\left\{\gamma^\mu \not{q}\, \gamma^\nu \not{q}\right\} + m^2 \operatorname{Tr}\left\{\gamma^\mu \gamma^\nu\right\} \Big\}\end{aligned}}$$

↳ DIRAC ALGEBRA IN D - DIMENSIONS

WHEN USING DIMENSIONAL REGULARIZATION WE ALSO NEED TO DEFINE DIRAC γ- MATRICES IN D DIMENSIONS

- $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \mathbb{1}$

  ↳ UNIT MATRIX IN $f(D)$ DIM WITH $f(D=4) = 4$

- $\gamma^\mu \gamma_\mu = D \, \mathbb{1}$

- $$\begin{aligned}\gamma^\alpha \gamma^\mu \gamma_\alpha &= \gamma^\alpha \left(2 g^\mu_\alpha - \gamma_\alpha \gamma^\mu\right) \\ &= (2 - D)\, \gamma^\mu.\end{aligned}$$

- $$\begin{aligned}\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha &= \gamma^\alpha \gamma^\mu \left(2 g^\nu_\alpha - \gamma_\alpha \gamma^\nu\right) \\ &= 2 \gamma^\nu \gamma^\mu - \gamma^\alpha \gamma^\mu \gamma_\alpha \gamma^\nu \\ &= 4 g^{\mu\nu} - 2 \gamma^\mu \gamma^\nu - (2 - D)\, \gamma^\mu \gamma^\nu \\ &= 4 g^{\mu\nu} - (4 - D)\, \gamma^\mu \gamma^\nu\end{aligned}$$

- γ - MATRICES ARE $f(D) \times f(D)$ MATRICES WITH $f(D=4) = 4$

$$f(D) = f(4) + f'(4)\,(D-4) + O(\varepsilon^2)$$
$$= 4 - 2\varepsilon\, f'(4) + O(\varepsilon^2)$$

WE WILL SEE THAT PHYSICAL OBSERVABLES WILL NOT DEPEND ON $f'(4)$

↳ CONVENTIONAL CHOICE $f'(4) = 0$

- $\mathrm{Tr}\, \mathbb{1} = f(D)$
- $\mathrm{Tr}\left\{ \gamma^\alpha \gamma^\beta \right\} = f(D)\, g^{\alpha\beta}$
- $\mathrm{Tr}\left\{ \gamma^\alpha \gamma^\beta \gamma^\kappa \gamma^\lambda \right\} = f(D) \left\{ g^{\alpha\beta} g^{\kappa\lambda} - g^{\alpha\kappa} g^{\beta\lambda} + g^{\alpha\lambda} g^{\beta\kappa} \right\}$
- $\mathrm{Tr}\left\{ \text{ODD \# } \gamma\text{- MATRICES} \right\} = 0.$

$$\hookrightarrow \quad Tr\left\{ \gamma^\mu \not{k} \gamma^\nu \not{k} \right\} - x(1-x)\, Tr\left\{ \gamma^\mu \not{q} \gamma^\nu \not{q} \right\} + m^2\, Tr\left\{ \gamma^\mu \gamma^\nu \right\}$$

$$= f(D) \left\{ 2k^\mu k^\nu - k^2 g^{\mu\nu} - x(1-x)\left[2q^\mu q^\nu - q^2 g^{\mu\nu} \right] + m^2 g^{\mu\nu} \right\}$$

↳ VACUUM POLARIZATION IN DIMENSIONAL REGULARIZATION

$$\Pi^{\mu\nu}(q) = i e^2 \mu^{2\varepsilon} \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 - \Delta + i\varepsilon\right]^2}$$

$$\cdot f(D) \left\{ 2k^\mu k^\nu - k^2 g^{\mu\nu} + g^{\mu\nu}\left(m^2 + q^2 x(1-x)\right) - x(1-x)\, 2 q^\mu q^\nu \right\}$$

WITH $\Delta = m^2 - q^2 x(1-x)$

$$\Pi^{\mu\nu}(q) = i e^2 \mu^{2\varepsilon} f(D) \frac{i}{(4\pi)^2} (4\pi)^\varepsilon$$

$$\cdot \int_0^1 dx \left\{ -g^{\mu\nu} \frac{(2-D)}{2}\, \Gamma\left(1-\frac{D}{2}\right) \cdot \frac{1}{\Delta^{1-D/2}} + \Gamma(\varepsilon) \frac{1}{\Delta^\varepsilon} \left[g^{\mu\nu}\left(m^2 + q^2 x(1-x)\right) - x(1-x)\, 2 q^\mu q^\nu \right] \right\}$$

$$\Pi^{\mu\nu}(q) = - \frac{e^2}{(4\pi)^2} (4\pi)^{\varepsilon} \mu^{2\varepsilon} f(D) \,.\, \Gamma(\varepsilon)$$

$$\cdot \int_0^1 dx\, \Delta^{-\varepsilon} \Big\{ - g^{\mu\nu} \left(\cancel{m}^2 - q^2 x(1-x) \right) + g^{\mu\nu} \left(\cancel{m}^2 + q^2 x (1-x) \right) - 2 q^\mu q^\nu x(1-x) \Big\}$$

$$\Pi^{\mu\nu}(q) = -\left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \frac{e^2}{2\pi^2}$$

$$\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln(4\pi) + O(\varepsilon) \right\}$$

$$\cdot \int_0^1 dx\, x(1-x) \left\{ 1 + \varepsilon \ln\left(\frac{\mu^2}{\Delta} \right) + O(\varepsilon^2) \right\}$$

$$\boxed{\begin{aligned} \Pi^{\mu\nu}(q) &= - \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \cdot \frac{e^2}{12\pi^2} \\ &\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln(4\pi) + 6 \int_0^1 dx\, x(1-x) \ln\left(\frac{\mu^2}{m^2 - q^2 x(1-x)} \right) \right\} \end{aligned}}$$

$$\boxed{\begin{aligned} \Pi^{\mu\nu}(q) &\overset{Q^2 \gg m^2}{\simeq} - \frac{e^2}{12\pi^2} \left(q^2 g^{\mu\nu} - q^\mu q^\nu \right) \\ &\cdot \left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2} f'(4) + \ln 4\pi + \ln \frac{\mu^2}{Q^2} + \frac{5}{3} \right\} \end{aligned}}$$

2) RUNNING COUPLING IN QED

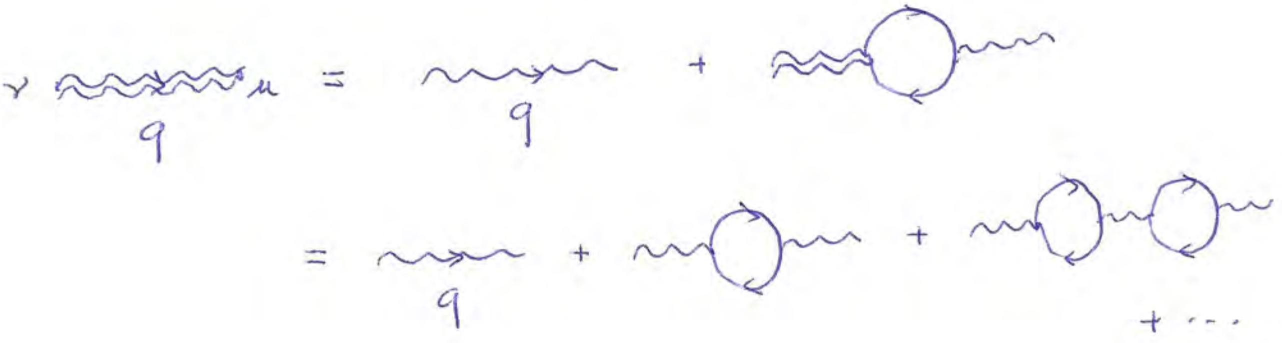
⇒ PHYSICAL INTERPRETATION OF VACUUM POLARIZATION IN QED

QED :

$$\hookrightarrow \quad \Pi^{\mu\nu}(q) = -\frac{e^2}{12\pi^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$\left\{ \frac{1}{\varepsilon} - \gamma_E - \frac{1}{2}\rho'(4) + \ln 4\pi + 6\int_0^1 dx \; x(1-x)\,\ln\left(\frac{\mu^2}{m^2 - q^2 x(1-x)}\right)\right\}$$

↳ 'DRESSED' PHOTON PROPAGATOR AT 1-LOOP IN FEYNMAN GAUGE (DYSON EXPANSION)



$$i\,D^{\mu\nu}(q) = i\,D_0^{\mu\nu}(q) + i\,D_0^{\mu\alpha}(q)\; i\,\Pi_{\alpha\beta}(q)\; i\,D^{\beta\nu}(q)$$

$$\text{WITH} \quad \Pi^{\alpha\beta}(q) \equiv \left(q^2 g^{\alpha\beta} - q^\alpha q^\beta\right)\underset{\substack{\uparrow \\ \text{SCALAR} \\ \text{FUNCTION}}}{\Pi(q^2)}$$

DYSON SERIES CORRESPONDS WITH RESUMMATION OF VACUUM POL.

$$D^{\mu\nu}(q) = D_0^{\mu\nu}(q) - D_0^{\mu\alpha}(q)\; \Pi_{\alpha\beta}(q)\; D_0^{\beta\nu}(q)$$

$$+ D_0^{\mu\alpha}(q)\; \Pi_{\alpha\beta}(q)\; D_0^{\beta\gamma}(q)\; \Pi_{\gamma\delta}(q)\; D_0^{\delta\nu}(q)$$

$$+ \cdots$$

$\downarrow$ IN FEYNMAN GAUGE $D_0^{\mu\nu} = - \dfrac{g^{\mu\nu}}{q^2}$

$$D^{\mu\nu}(q) = - \frac{g^{\mu\nu}}{q^2} + \frac{-g^{\mu\alpha}}{q^2}\; \Delta_\alpha{}^\nu(q)\; \Pi(q^2)$$

$$+ \frac{-g^{\mu\alpha}}{q^2}\; \Delta_\alpha{}^\gamma(q)\; \Delta_\gamma{}^\nu(q)\; \Pi^2(q^2)$$

$$+ \cdots$$

WHERE WE HAVE DEFINED

$$\Pi_{\alpha\beta}(q) \equiv \Delta_{\alpha\beta}(q)\; q^2\, \Pi(q^2)$$

$$\Delta_{\alpha\beta}(q) \equiv g_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}$$

NOTE $\Delta_\alpha{}^\gamma(q)\, \Delta_\gamma{}^\nu(q) = \Delta_\alpha{}^\nu(q)$

$$D^{\mu\nu}(q) = -\frac{g^{\mu\nu}}{q^2} - \frac{1}{q^2}\,\Delta^{\mu\nu}(q)\left[\Pi(q^2) + \Pi^2(q^2) + \cdots\right]$$

$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)\frac{1}{q^2}\left[1 + \Pi(q^2) + \Pi^2(q^2) + \cdots\right]$$

$$- \frac{1}{q^2}\,\frac{q^\mu q^\nu}{q^2}$$

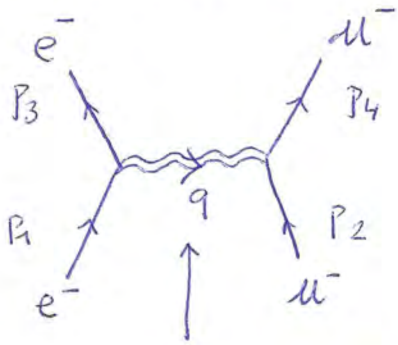
$\downarrow$ GEOMETRIC SERIES

$$\boxed{D^{\mu\nu}(q) = \frac{1}{q^2\left[1 - \Pi(q^2)\right]}\cdot\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) - \frac{1}{q^2}\,\frac{q^\mu q^\nu}{q^2}}$$

WITH VACUUM POL. TO 1 LOOP

$$\boxed{\Pi(q^2) = -\frac{e^2}{12\pi^2}\left\{\frac{1}{\varepsilon} - \gamma_E - \frac{1}{2}\,\zeta'(4) + \ln 4\pi + 6\int_0^1 dx\; x(1-x)\,\ln\left(\frac{\mu^2}{m^2 - q^2 x(1-x)}\right)\right\}}$$

↳ PHYSICAL INTERPRETATION (e.g. $e^-\mu^-$ SCATTERING)



FULL PROPAGATOR

$$iD^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2\left[1-\Pi(q^2)\right]} + q^\mu q^\nu \text{ TERMS}$$

NOTE : $q^\mu q^\nu$ TERMS GIVE NO CONTRIBUTION TO PHYSICAL AMPLITUDES WHEN COUPLED TO CONSERVED CURRENTS

TRANSITION AMPLITUDE

$$\mathcal{M} = \bar{u}(p_3,s_3)\left(ie\gamma_\nu\right)u(p_1,s_1) \quad iD^{\mu\nu}(q)$$
$$\cdot\ \bar{u}(p_4,s_4)\left(ie\gamma_\mu\right)u(p_2,s_2)$$

AS $$q^\nu\ \bar{u}(3)\gamma_\nu u(1) = \bar{u}(3)\left(\not{p}_1-\not{p}_3\right)u(1) = 0$$
$$q^\mu\ \bar{u}(4)\gamma_\mu u(2) = \bar{u}(4)\left(\not{p}_4-\not{p}_2\right)u(2) = 0$$

↓

$$\boxed{\mathcal{M} = ie^2\ \bar{u}(3)\gamma^\mu u(1)\cdot \bar{u}(4)\gamma_\mu u(1)\cdot \frac{1}{q^2\left[1-\Pi(q^2)\right]}}$$

COULOMB POTENTIAL IS MODIFIED DUE TO VACUUM POL

↳ TO LOWEST ORDER $V_c(q^2) \equiv \dfrac{e^2}{q^2}$

↳ INCLUDING VACUUM POL $V(q^2) = \dfrac{e^2}{q^2\left[1-\Pi(q^2)\right]}$

IF $\Pi(q^2)$ IS A REGULAR AT $q^2=0$

$$\Pi(q^2) = \Pi(0) + q^2\,\Pi'(0) + \cdots$$

$$\boxed{V(q^2) = \frac{e^2}{q^2\left(1-\Pi(0)\right)} \cdot \left\{1 + \frac{\Pi'(0)}{1-\Pi(0)}\,q^2 + O(e^4)\right\}}$$

∴ FULL PROPAGATOR STILL HAS POLE AT $q^2=0$ (PHOTON REMAINS MASSLESS)

↳ RESIDUE OF POLE $\boxed{\dfrac{1}{1-\Pi(0)} \equiv Z_3}$

$$\boxed{\begin{aligned} Z_3 &= 1 + \Pi(0) + O(e^4) \\ &= 1 - \frac{e^2}{12\pi^2}\left\{\frac{1}{\varepsilon} - \gamma_E - \frac{1}{2}f'(4) + \ln\frac{4\pi\mu^2}{m^2}\right\} \\ &\quad + O(e^4) \end{aligned}}$$

⇒ CHARGE RENORMALIZATION IN QED

↳ DEFINE PHYSICAL (RENORMALIZED) CHARGE AT REFERENCE SCALE (i.e. $q^2 = 0$ CORRESPONDING TO $r = \infty$ IN COORDINATE SPACE)

$$\boxed{e_R^{\,2} \equiv e^2 Z_3}$$

INFINITY IN VACUUM POL. IS ABSORBED IN RE-DEFINITION OF CHARGE (Z_3 : RENORMALIZATION CONSTANT)

⇝ e_R^2 IS PHYSICAL (FINITE) CHARGE MEASURED IN EXPERIMENT

⇝ e^2 IS CALLED 'BARE' (UNPHYSICAL) CHARGE

↳ POTENTIAL IN TERMS OF e_R^2

$$V(q^2) = \frac{e^2}{q^2\left[1 - \Pi(0) - \left(\Pi(q^2) - \Pi(0)\right)\right]}$$

$$= \frac{e^2}{q^2\left[1-\Pi(0)\right]}\left\{1 + \frac{\Pi(q^2) - \Pi(0)}{1 - \Pi(0)} + O(e^4)\right\}$$

$$\boxed{V(q^2) = \frac{e_R^2}{q^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx\, x(1-x) \ln\left[1 - \frac{q^2}{m^2}x(1-x)\right] + O(e_R^4)\right\}}$$

↳ POTENTIAL IN LIMIT $Q^2 = -q^2 \gg m^2$

FOR SCATTERING PROCESS $q^2 < 0 \Rightarrow Q^2 \equiv -q^2 > 0$

CONSIDER LIMIT $Q^2 \gg m^2$

$$\ln\left[1 + \frac{Q^2}{m^2} x(1-x)\right] = \ln\frac{Q^2}{m^2} + \ln\left[x(1-x) + \frac{m^2}{Q^2}\right]$$

$$\underset{Q^2 \gg m^2}{\approx} \ln\frac{Q^2}{m^2}$$

$$V(q^2) = \frac{e_R^2}{q^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx\; x(1-x) \ln\frac{Q^2}{m^2} + O(e_R^4)\right\}$$

$$\boxed{V(q^2) \simeq \frac{1}{q^2} e_R^2 \left\{1 + \frac{e_R^2}{12\pi^2} \ln\frac{Q^2}{m^2} + O(e_R^4)\right\}}$$

↳ EFFECTIVE CHARGE / RUNNING COUPLING

$$\boxed{V(q^2) \equiv \frac{e_{eff}^2(Q^2)}{q^2}}$$

AT SCALE $q^2 = -Q^2$

POTENTIAL LOOKS LIKE COULOMB POT. WITH
EFFECTIVE q^2 DEPENDENT CHARGE

$$\boxed{e_{eff}^2(Q^2) \equiv e_R^2 \left\{ 1 + \frac{e_R^2}{12\pi^2} \ln\frac{Q^2}{m^2} + O(e_R^4) \right\}}$$

NOTE : e_{eff}^2 INCREASES FOR $Q^2 \uparrow$
CORRESPONDING WITH SHORTER
DISTANCE SCALES

(AT LARGE DISTANCES: CHARGE IS MORE AND
MORE SCREENED BY VIRTUAL e^-e^+ DIPOLES
$\Rightarrow$ DECREASES AT LARGE DISTANCES.

WE CAN TAKE RESUMMATION OF 1-LOOP INTO ACCOUNT AS

$$\boxed{e_{eff}^2(Q^2) = \frac{e_R^2}{1 - \frac{e_R^2}{12\pi^2} \ln\frac{Q^2}{m^2}} + O(e_R^4)}$$

OR BY DEFINING $\alpha_{em}(Q^2) \equiv \dfrac{e_{eff}^2(Q^2)}{4\pi}$

$$\alpha_{em}(m^2) \equiv \frac{e_{eff}^2(m^2)}{4\pi}$$

$$\boxed{\alpha_{em}(Q^2) = \frac{\alpha_{em}(m^2)}{1 - \frac{\alpha_{em}(m^2)}{3\pi} \ln\frac{Q^2}{m^2}}}$$

RUNNING COUPLING IN QED TO 1-LOOP

NOTE :
- ONE FIXES CHARGE AT LARGE DISTANCE TO EXPERIMENT

$$\alpha_{em}(m^2) \simeq \frac{1}{137} \quad \text{FINE-STRUCTURE CONSTANT}$$

- THEN EFFECTIVE CHARGE AT ANY SCALE CAN BE PREDICTED THROUGH

$$\alpha_{em}(Q^2) = \frac{\alpha_{em}(m^2)}{1 - \frac{\alpha_{em}(m^2)}{3\pi} \ln \frac{Q^2}{m^2}}$$

- EXAMPLE

FOR $Q^2 \simeq (90 \text{ GeV})^2$ NEAR Z^0 POLE

$$\ln\left(\frac{Q^2}{m^2}\right) \simeq 24$$

$$\frac{\alpha_{em}(m^2)}{3\pi} \cdot \ln \frac{Q^2}{m^2} \simeq 0,019$$

($\approx 2\%$ CORRECTION)

- VERY SLOW (LOGARITHMIC RISE) OF COUPLING
- AT VERY HIGH SCALE

$$\frac{\alpha_{em}(m^2)}{3\pi} \ln \frac{Q^2}{m^2} \approx 1 \Longleftrightarrow Q \approx 10^{286} \text{ eV}$$

CORRECTION AS LARGE AS LEADING TERM (LANDAU POLE) → PERTURBATION THEORY BREAKS DOWN

⟹ UEHLING POTENTIAL / LAMB SHIFT

↳ CONSIDER EFFECTIVE 1-LOOP POTENTIAL

$$V(q^2) = \frac{e_R^2}{q^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ln\left[1-\frac{q^2}{m^2}x(1-x)\right] + O(e_R^4)\right\}$$

IN LARGE DISTANCE LIMIT $\underset{\sim}{q^2 \ll m^2}$

$$V(q^2) \underset{q^2\ll m^2}{\simeq} \frac{e_R^2}{q^2}\left\{1 - \frac{e_R^2}{2\pi^2}\frac{q^2}{m^2}\int_0^1 dx\; x^2(1-x)^2 + O(e_R^4)\right\}$$

$$= \frac{e_R^2}{q^2} - \frac{e_R^4}{60\pi^2 m^2} + O(e_R^6)$$

↳ POTENTIAL IN COORDINATE SPACE $(q^2 = -\bar{q}^2)$

$$V(r) = \int\frac{d^3\bar{q}}{(2\pi)^3}\, e^{i\bar{q}\cdot\bar{r}}\, V(\bar{q}^2)$$

$$\boxed{V(r) = \underset{\uparrow}{\frac{-e_R^2}{4\pi r}} - \underset{\Uparrow}{\frac{e_R^4}{60\pi^2 m^2}}\,\delta^3(\bar{r}) + O(e_R^6)}$$

COULOMB TERM   UEHLING POTENTIAL.

$$\frac{e_R^2}{4\pi} = \alpha_{em} \simeq \frac{1}{137}$$ FINE-STRUCTURE CONSTANT

$$V(r) \simeq -\frac{\alpha_{em}}{r} - \frac{4}{15}\frac{\alpha_{em}^2}{m^2}\delta^3(\bar{r}) + O(\alpha_{em}^3)$$

↳ LAMB SHIFT IN HYDROGEN

- ENERGY LEVELS IN HYDROGEN ATOM

$$E_n = \int d^3\bar{r}\ \psi_n^* \left(-\frac{\alpha_{em}}{r}\right)\psi_n = -\frac{m\alpha_{em}^2}{2}\frac{1}{n^2}$$

(BOHR)

- CORRECTION DUE TO UEHLING TERM

$\delta^3(\bar{r}) \longrightarrow$ ONLY S-LEVELS ARE AFFECTED

ΔE_{nS}^{VP}: CORRECTION TO nS LEVEL DUE TO VACUUM POLARIZATION

$$\Delta E_{nS}^{VP} = -\frac{4}{15}\frac{\alpha_{em}^2}{m^2}\underbrace{|\psi_{nS}(0)|^2}_{\text{WAVE FUNCTION AT ORIGIN}}$$

$$\Big\downarrow \quad |\psi_{nS}(0)|^2 = \frac{m^3\alpha_{em}^3}{\pi n^3}$$

$$\boxed{\Delta E_{nS}^{VP} = -\frac{4}{15\pi}\frac{m\alpha_{em}^5}{n^3}}$$

LAMB SHIFT CONTRIBUTION DUE TO VP

- FOR $n = 2$

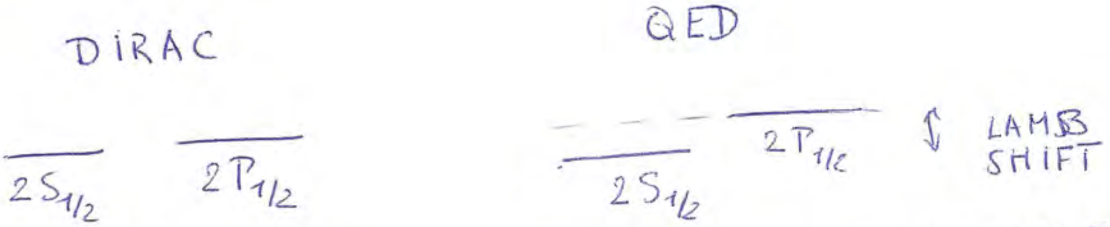
$$\Delta E_{2S} = -\frac{m \alpha_{em}^5}{30 \pi} = -1.1 \; 10^{-7} \, eV \; (H)$$

- IN DIRAC THEORY

$2S_{1/2}$ & $2P_{1/2}$ ARE DEGENERATE

DUE TO LOOP CORRECTIONS IN QED

DEGENERACY IS LIFTED



VACUUM POL IS ONE CONTRIBUTION TO LAMB SHIFT

$$\Delta E^{\text{VAC POL}} = E(2P_{1/2}) - E(2S_{1/2})$$

$$= \frac{m \alpha_{em}^5}{30\pi}$$

$$\equiv \hbar\omega = h\nu$$

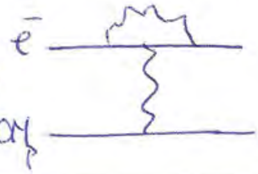
FREQUENCY $\nu_{\substack{VAC\\POL}} = \frac{c}{2\pi \hbar c} \hbar\omega$

$$= \frac{3. \, 10^8 \, m/s \quad (1.1 \; 10^{-7} eV)}{2\pi \; (197) \; 10^{-9} \, eV \, m}$$

$$= \underline{\underline{27 \text{ MHz}}}$$

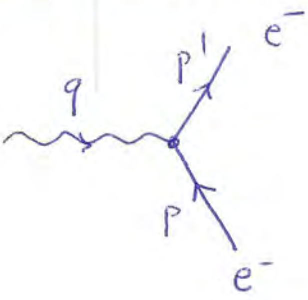
- EXP. LAMB SHIFT (1947)

$\nu_{LS} \simeq 1000$ MHz : DOMINATED BY VERTEX CORRECTION



3) ANOMALOUS MAGNETIC MOMENT IN QED

⇒ LOWEST ORDER VERTEX

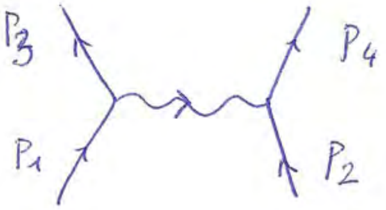


- FEYNMAN RULE

$$\mathcal{M}^{\mu}_{(0)}(p',p) \equiv \bar{u}(p',s')\,(ie\gamma^{\mu})\,u(p,s)$$

↑
LOWEST ORDER

THIS ENTERS e.g. PHYSICAL PROCESSES AS



$$\mathcal{M}_{(0)} = \mathcal{M}^{\mu}_{(0)}(p_3,p_1)\left(-\frac{i g_{\mu\nu}}{q^2}\right)\mathcal{M}^{\nu}_{(0)}(p_4,p_1)$$

↑
INVARIANT AMPLITUDE

- GORDON DECOMPOSITION

$$\bar{u}(p',s')\ \gamma^{\mu}\ u(p,s)$$

$$= \bar{u}(p',s')\left\{\frac{(p+p')^{\mu}}{2m} + i\sigma^{\mu\nu}\frac{q_{\nu}}{2m}\right\} u(p,s)$$

WITH $q = p' - p$ (MOMENTUM TRANSFER TOWARDS VERTEX)

→ $(p+p')^{\mu}$ TERM : CONVECTION CURRENT

→ $i\sigma^{\mu\nu}q_{\nu}$ TERM : MAGNETIZATION CURRENT

RESPONSIBLE FOR $g = 2$
OF DIRAC THEORY

↓

COEFFICIENT IN FRONT OF $i\sigma^{\mu\nu}\frac{q_{\nu}}{2m}$ TERM

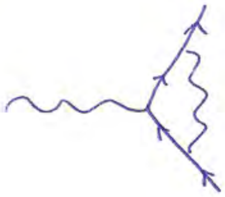
IS $\frac{g}{2}$ → IN DIRAC THEORY $\frac{g_{DIRAC}}{2} = \underline{1}$.

⟹ GENERAL VERTEX FOR SPIN 1/2

- DUE TO LOOP CORRECTIONS, WE WILL GET A DIFFERENCE FROM DIRAC RESULT

  ANOMALOUS MAGNETIC MOMENT = DEVIATION FROM DIRAC THEORY

$$\boxed{a \equiv \frac{g-2}{2}}$$

WE WILL CALCULATE 

- IN GENERAL

$$\mathcal{M}^{\mu}(p', p) \equiv \bar{u}(p', s')\ (ie\,\Gamma^{\mu})\ u(p, s)$$

↳ Γ^{μ} IS A 4-VECTOR

INDEPENDENT 4-VECTORS $q^{\mu}, (p+p')^{\mu}$

$\bar{u}\,\gamma^{\mu}\,u$

<u>NOTE</u>: $\bar{u}\,\sigma^{\mu\nu} q_{\nu}\, u$ IS NOT INDEPENDENT AS IT CAN BE REDUCED TO ABOVE 3 USING GORDON ID

BASED ON LORENTZ COVARIANCE

$$\mathcal{M}^{\mu}(p',p) = \bar{u}(p',s')\left\{ a\,\gamma^{\mu} + b\,(p+p')^{\mu} + c\,q^{\mu} \right\} u(p,s)$$

↳ GAUGE INVARIANCE

$\mathcal{M}^{\mu}$ IS A CONSERVED CURRENT

$$q_{\mu}\mathcal{M}^{\mu} = 0$$

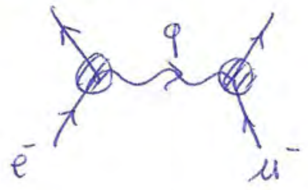
- $\bar{u}(p')\,\not{q}\;u(p) = \bar{u}(p')\,(\not{p}' - \not{p})\,u(p) = \bar{u}(p')\,(m-m)\,u(p) \overset{!}{=} 0$

- $\bar{u}(p')\,q\cdot(p+p')\,u(p) = \bar{u}(p')\,(p'^2 - p^2)\,u(p) = \bar{u}(p')\,(m^2-m^2)\,u(p) \overset{!}{=} 0$

  ↑ ON-SHELL e^- INITIAL & FINAL

- $\bar{u}(p')\;q\cdot q\;u(p) = \bar{u}(p')\,q^2\,u(p) \neq 0$

  BECAUSE γ IS IN GENERAL OFF-SHELL IN A PROCESS AS



$q^2 < 0$

∴ TO ENSURE GAUGE INVARIANCE WE NEED $c = 0$

↳ WE CAN RE-EXPRESS $(p+p')^\mu$ TERM IN TERMS OF γ^μ AND $\sigma^{\mu\nu} q_\nu$ TERMS USING GORDON ID

MOST GENERAL VERTEX FOR SPIN 1/2

$$\boxed{\mathcal{M}^\mu(p',p) = ie\bar{u}(p',s')\left\{ F_1(q^2)\, \gamma^\mu + F_2(q^2)\, i\sigma^{\mu\nu}\frac{q_\nu}{2m} \right\} u(p,s)}$$

NOTE : COEFF IN FRONT OF γ^μ & $i\sigma^{\mu\nu}q_\nu$ ARE SCALAR FUNCTIONS

CAN DEPEND ON ALL INDEP. SCALARS

$q^2,\ q\cdot p\ ,\ p^2$

BUT $p^2 = m^2$ CONSTANT

$p'^2 = m^2 = (p+q)^2 \Rightarrow 2q\cdot p = -q^2$

$\therefore$ ONLY q^2 IS INDEPENDENT SCALAR

$F_1(q^2)$, $F_2(q^2)$ ARE CALLED 'FORM FACTORS' (FF)

$F_1(q^2)$: DIRAC FF

$F_2(q^2)$: PAULI FF

↳ $F_1(q^2=0) = 1.$ CHARGE IN UNITS e

↳ $F_2(q^2=0) = a$ ANOMALOUS MAGNETIC MOMENT

TOTAL MAGNETIC MOMENT

$$\bar{\mu} = \frac{e}{2m}\left(\underset{\uparrow}{1} + a\right)\underset{\uparrow}{\bar{\sigma}}$$

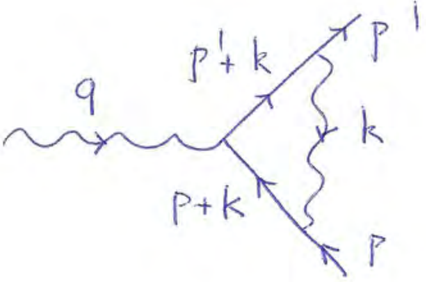
PAULI MATRICES

DIRAC
VALUE
$(g=2)$

NOTE : $g = 2(1+a)$

⇒ ONE-LOOP CORRECTION TO a

WE WILL CALCULATE 1-LOOP CORRECTION TO ANOMALOUS MAGNETIC MOMENT OF ELECTRON (FERMION) DUE TO VERTEX CORRECTION



- $\mathcal{M}^{\mu}_{(1)}(p',p) = ie\,\bar{u}(p',s')$

↑ 1-LOOP

$$\cdot\int\frac{d^4k}{(2\pi)^4}\frac{(ie\gamma^{\alpha})\,i(\not p'+\not k+m)(-ig_{\alpha\beta})\,\gamma^{\mu}\,i(\not p+\not k+m)(ie\gamma^{\beta})}{\left[(p'+k)^2-m^2+i\varepsilon\right]\left[(p+k)^2-m^2+i\varepsilon\right]\left[k^2+i\varepsilon\right]}$$

$$\cdot\ u(p,s)$$

$p^2 = p'^2 = m^2$

$$= ie\,\bar{u}(p',s')$$

$$\cdot(-ie^2)\int\frac{d^4k}{(2\pi)^4}\frac{(2p'^{\alpha}+\gamma^{\alpha}\not k)\,\gamma^{\mu}\,(2p_{\alpha}+\not k\gamma_{\alpha})}{(k^2+2k\cdot p'+i\varepsilon)(k^2+2k\cdot p+i\varepsilon)(k^2+i\varepsilon)}\,u(p,s)$$

<u>NOTE</u>: WE HAVE USED

$$\bar{u}(p',s')\,\gamma^{\alpha}(\not p'+m)\ldots = \bar{u}(p',s')(2p'^{\alpha})\ldots$$

$$\ldots(\not p+m)\,\gamma_{\alpha}\,u(p,s) = \ldots(2p_{\alpha})\,u(p,s)$$

- FEYNMAN PARAMETERIZATION

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^x dy \; \frac{1}{\left[A + (B-A)x + (C-B)y\right]^3}$$

$$A = k^2 - m^2$$

$$B = k^2 + 2k.p$$

$$C = k^2 + 2k.p'$$

$$\mathcal{M}^{\mu}_{(1)}(p',p) = ie\,\bar{u}(p',s')$$

$$\cdot(-ie^2)\cdot\int\frac{d^4k}{(2\pi)^4}\, 2\int_0^1 dx\int_0^x dy\,\frac{\left(2p'^{\alpha}+\gamma^{\alpha}\not{k}\right)\gamma^{\mu}\left(2p_{\alpha}+\not{k}\gamma_{\alpha}\right)}{\left[k^2-m^2+\left(2k.p+m^2\right)x+2k.q\,y\right]^3}$$

$$\cdot\; u(p,s)$$

$$k' = k + px + qy$$

$$= k + p(x-y) + p'y$$

$$= k + p'x + q(y-x)$$

$$k^2 + 2k.p\,x + 2k.q\,y$$

$$= (k+px+qy)^2 - (px+qy)^2$$

$$= k'^2 - m^2x^2 - q^2y^2 - \underbrace{2p.q}_{-q^2}\,xy$$

$$= k'^2 - m^2x^2 + q^2y(x-y)$$

$$\mathcal{M}^{\mu}_{(1)}(p',p) = ie\,\bar{u}(p',s')$$

$$\cdot (-ie^2)\, 2 \int_0^1 dx \int_0^x dy \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 - \Delta + i\varepsilon\right]^3}$$

$$\cdot \left(2p'^{\alpha}(1-x) + \gamma^{\alpha} m x - \gamma^{\alpha} \not{q}\,(y-x) + \gamma^{\alpha} \not{k} \right) \cdot \gamma^{\mu}$$

$$\cdot \left(2p_{\alpha}(1-x) + \gamma_{\alpha} m x - \not{q}\, y\, \gamma_{\alpha} + \not{k}\, \gamma_{\alpha} \right) u(p,s)$$

NOTE 1) BEHAVIOR OF INTEGRAND FOR $k \to \infty$

$$d^4k \; \frac{1}{k^6} \cdot k^2 = \frac{d^4k}{k^4}$$ LOGARITHMIC DIVERGENCE

$\downarrow$

WE USE DIM REG FOR DIVERGENT TERM $\sim k^2$

FINITE TERM : IND. OF k IN NUMERATOR CAN BE EVALUATED IN $D=4$

TERMS LINEAR IN k $\rightarrow$ GIVE 0 UPON $\int d^4k \ldots$

2)
$$\boxed{\Delta \equiv m^2 x^2 - q^2 y (x-y)}$$

- NUMERATOR

$$\bar{u}(p',s')\left\{\left(2p'^{\alpha}(1-x)+\gamma^{\alpha}mx-\gamma^{\alpha}\not{q}\,(y-x)\right)\gamma^{\mu}\right.$$

$$\cdot\left(2p_{\alpha}(1-x)+\gamma_{\alpha}mx-\not{q}\,y\,\gamma_{\alpha}\right)$$

$$\left.+\ \gamma^{\alpha}\not{k}\,\gamma^{\mu}\not{k}\,\gamma_{\alpha}\right\}\ u(p,s)$$

⤳ FOR 2° TERM : RULE IN D DIM

$$\gamma^{\alpha}\not{k}\,\gamma^{\mu}\not{k}\,\gamma_{\alpha} = -2\not{k}\,\gamma^{\mu}\not{k} + 2\varepsilon\,\not{k}\,\gamma^{\mu}\not{k}$$

$$= -2(1-\varepsilon)\not{k}\,\gamma^{\mu}\not{k}$$

⤳ FOR 1° TERM : RULES IN 4 DIM

$$\gamma^{\alpha}\gamma_{\alpha} = 4$$

$$\gamma^{\alpha}\gamma^{\mu}\gamma_{\alpha} = -2\gamma^{\mu}$$

$$\gamma^{\alpha}\gamma^{\mu}\gamma^{\nu}\gamma_{\alpha} = 4g^{\mu\nu}$$

$$\gamma^{\alpha}\gamma^{\kappa}\gamma^{\lambda}\gamma^{\mu}\gamma_{\alpha} = -2\gamma^{\mu}\gamma^{\lambda}\gamma^{\kappa}$$

NUMERATOR

$$= \bar{u}(p',s')\Big\{\, 4p\cdot p'(1-x)^2\gamma^\mu + 2mx(1-x)\overbrace{\gamma^\mu \not{p}'}^{2p'^\mu - m\gamma^\mu}$$

$$-\,2m^2x^2\,\gamma^\mu + 2mx(1-x)\underbrace{\not{p}\,\gamma^\mu}_{2p^\mu - m\gamma^\mu}$$

$$-\,2y(y-x)\,\not{q}\underbrace{\gamma^\mu\not{q}}_{2q^\mu - \not{q}\gamma^\mu}$$

$$-\,2y(1-x)\,\gamma^\mu\not{q}\,\not{p}' - 2(y-x)(1-x)\,\not{p}\,\not{q}\,\gamma^\mu$$

$$-\,4m\,yx\,q^\mu - 4m(y-x)x\,q^\mu$$

$$-\,2(1-\varepsilon)\,\not{k}\,\gamma^\mu\not{k}\,\Big\}\,u(p,s)$$

$$= \bar{u}(p',s')\Big\{\, 2(2m^2-q^2)(1-x)^2\gamma^\mu + 4mx(1-x)(p+p')^\mu$$

$$-\,4m^2x(1-x)\gamma^\mu - 2m^2x^2\gamma^\mu$$

$$-\,\cancel{4y(y-x)q^\mu\not{q}} \searrow 0 + 2y(y-x)q^2\gamma^\mu$$

$$-\,4m\,x(2y-x)q^\mu - 2(1-\varepsilon)\,\not{k}\,\gamma^\mu\not{k}$$

$$-\,2y(1-x)\left[2m\,p'^\mu + (q^2-2m^2)\gamma^\mu\right]$$

$$+\,2(y-x)(1-x)\left[2m\,p^\mu + (q^2-2m^2)\gamma^\mu\right]\Big\}\,u(p,s)$$

USING • $\bar{u}\gamma^\mu\not{q}\,\not{p}'u = \bar{u}\gamma^\mu(m^2-\not{p}\,\not{p}')u = \bar{u}\gamma^\mu(m^2-2p\cdot p' + m\not{p}')u$

$$= \bar{u}\left\{2m\,p'^\mu + (q^2-2m^2)\gamma^\mu\right\}u$$

• $\bar{u}\,\not{p}\,\not{q}\,\gamma^\mu u = \bar{u}\left\{-2m\,p^\mu - (q^2-2m^2)\gamma^\mu\right\}u$

$$\downarrow \quad \begin{aligned} 2p^\mu &= (p+p')^\mu - q^\mu \\ 2p'^\mu &= (p+p')^\mu + q^\mu \end{aligned}$$

OUR EXPRESSION HAS TERMS

$$\sim \gamma^\mu \, , \; (p+p')^\mu \, , \; q^\mu$$

- TERMS IN q^μ :

$$-4m\,x(2y-x) - 2m\,y(1-x) - 2m\,(y-x)(1-x)$$

$$= -4m\,x\,(2y-x) - 2m\,(1-x)(2y-x)$$

$$= -2m\,(1+x)(2y-x)$$

NOTE $\displaystyle\int_0^1 dx \int_0^x dy \, \frac{(2y-x)}{[\cdots q^2 y(x-y)]^3}$

$$= \int_0^1 dx \left\{ \int_0^{x/2} dy + \int_{x/2}^{x} dy \right\} \frac{(2y-x)}{[\cdots q^2 y(x-y)]^3}$$

$\downarrow$ IN 2° INT $y' = x - y$

$$\int_{x/2}^{x} dy \, \frac{(2y-x)}{[\cdots q^2 y(x-y)]^3} = \int_0^{x/2} dy' \, \frac{\overbrace{(x-2y')}^{(-1)(2y'-x)}}{[\cdots q^2 (x-y')y']^3}$$

2 INTEGRALS ARE EXACTLY OPPOSITE

∴ q^μ TERMS CANCEL ! AS REQUIRED BY GAUGE INVARIANCE

- **$F_2(q^2)$ TERM**

BY USING GORDON ID.,
WE CAN IDENTIFY $F_2(q^2)$ TERM
AS TERM PROPORTIONAL TO $-\frac{(p+p')^\mu}{2m}$

$$F_2(q^2) = -ie^2\, 2 \int_0^1 dx \int_0^x dy \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - \Delta + i\varepsilon]^3}$$

$$\cdot (-2m) \left\{ 4m\, x(1-x) - 2m\, y(1-x) + 2m(y-x)(1-x) \right\}$$

$$= ie^2\, 8m^2 \int_0^1 dx \int_0^x dy \int \frac{d^4k}{(2\pi)^4} \frac{x(1-x)}{[k^2 - \Delta + i\varepsilon]^3}$$

$$\boxed{F_2(q^2) = ie^2\, 8m^2 \int_0^1 dx\; x(1-x) \int_0^x dy\, \frac{i}{(4\pi)^2} \frac{(-1)}{2} \frac{1}{[m^2x^2 - q^2 y(x-y)]}}$$

- **$F_2(0) = a$**

$$a = F_2(0) = \frac{e^2}{4\pi^2} \cdot m^2 \int_0^1 dx\; x(1-x) \int_0^x dy\, \frac{1}{m^2x^2}$$

$$= \frac{\alpha_{em}}{\pi} \cdot \underbrace{\int_0^1 dx\, (1-x)}_{1/2}$$

$$\boxed{a = \frac{\alpha_{em}}{2\pi}}$$ SCHWINGER (1948)

⇒ STATUS OF a_μ : MUON $(g-2)$

- $$a_\mu^{(1)\to 1\text{ LOOP}} = \frac{\alpha_{em}}{2\pi} \simeq 1.16\ 10^{-3}$$

$$g = 2(1+a)$$

$$g^{(1)} = 2.0023\ldots$$

- PRESENT THEORETICAL VALUE

QED : CALCULATED UP TO 5 LOOPS !

~ 12 000 DIAGRAMS

ELECTROWEAK

QCD : LARGEST UNCERTAITNTY AT PRESENT !

$$a_\mu^{\text{THEORY}} = (11\,659\,182.8 \pm 4.9)\ 10^{-10}$$

- COMPARISON WITH EXPERIMENT (BROOKHAVEN 2004)

$$a_\mu^{\text{EXP}} = (11\,659\,208.9 \pm 6.3)\ 10^{-10}$$

- DIFFERENCE EXP − TH

$$a_\mu^{\text{EXP}} - a_\mu^{\text{TH}} = (26.1 \pm 8.0)\ 10^{-10}$$

~ 4σ DEVIATION ! (2017)

SIGNAL NEW PHYSICS ?

NEW EXPERIMENT IS ONGOING AT FERMILAB (USA)

↳ WILL IMPROVE EXP. ERROR FROM $6.3\ 10^{-10} \to 1.6\ 10^{-10}$

4) ELECTRON SELF-ENERGY IN QED

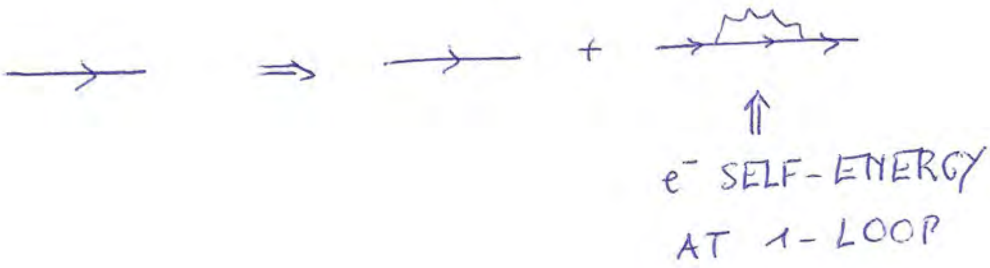
⇒ FERMION PROPAGATOR

↳ FREE FERMION PROPAGATOR

$$iS_0(p) = \frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$$

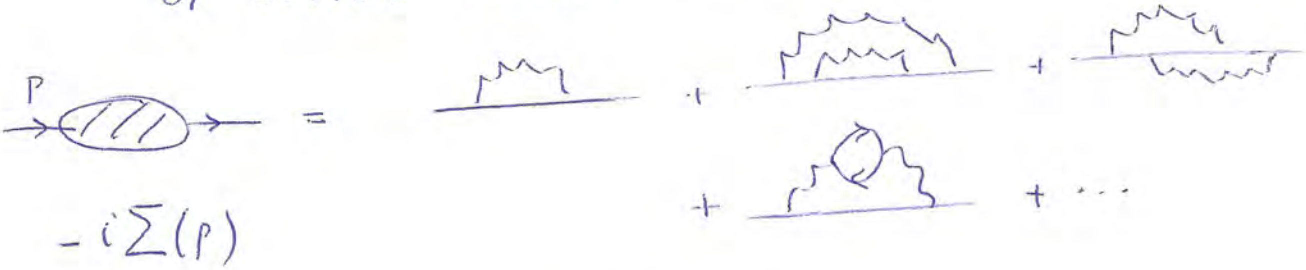
↳ FULL FERMION PROPAGATOR

- DUE TO 1-LOOP CORRECTION

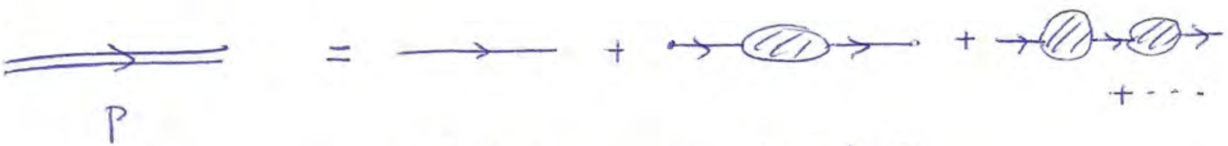


- DEFINE e⁻ SELF-ENERGY $-i\Sigma(p)$ AS ALL 1-FERMION IRREDUCIBLE CORRECTION GRAPHS

  i.e DO NOT BECOME DISCONNECTED BY CUTTING 1 FERMION LINE



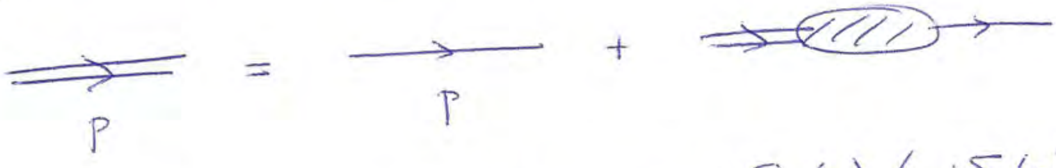
- FULL FERMION PROPAGATOR



$$iS(p) = iS_0(p) + iS_0(-i\Sigma)iS_0 + iS_0(-i\Sigma)iS_0(-i\Sigma)iS_0 + \dots$$

WE CAN RESUM THE 1 - FERMION REDUCIBLE GRAPHS IN A DYSON EQUATION

$\downarrow$



$$iS(p) = iS_0(p) + iS_0(p)\left(-i\Sigma(p)\right) iS(p)$$

$\downarrow$ MULTIPLY ON LEFT BY $S_0^{-1}(p) = \not{p} - m$

$$S_0^{-1}(p)\, S(p) = 1 + \Sigma(p)\, S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

$$\boxed{S(p) = \frac{1}{\not{p} - m - \Sigma(p)}}$$

- EXPANSION OF SELF-ENERGY (MATRIX IN DIRAC SPACE)

$$\boxed{\Sigma(p) = \not{p}\, \delta_2(p^2) + m\, \delta_m(p^2)}$$

δ_2 & δ_m ARE SCALAR FUNCTIONS OF p^2

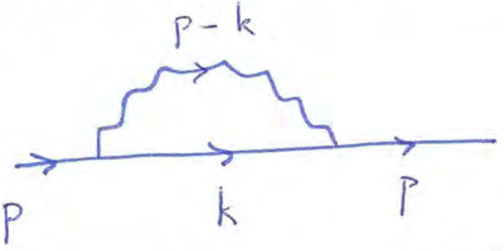
- $$\boxed{S(p) = \frac{1}{\left(1 - \delta_2(p^2)\right)\not{p} - m\left(1 + \delta_m(p^2)\right)}}$$

δ_2 : WILL RENORMALIZE FERMION FIELD

δ_m : WILL RENORMALIZE FERMION MASS

⇒ ELECTRON SELF-ENERGY IN QED TO 1-LOOP

↳



DENOTE ELECTRON MASS AS m_B : 'BARE' MASS, DEFINED IN ABSENCE OF INTERACTIONS

TO 1-LOOP : $O(e^2)$

$$-i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} \frac{(ie\gamma^\mu)\, i(\not{k}+m_B)(ie\gamma^\nu)(-ig_{\mu\nu})}{\left[k^2-m_B^2+i\varepsilon\right]\left[(p-k)^2+i\varepsilon\right]}$$

$\downarrow$ FEYNMAN PARAMETERIZATION

$$\Sigma(p) = -ie^2\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+m_B)\gamma_\mu}{\left[k^2-m_B^2+(p^2+m_B^2-2p.k)x+i\varepsilon\right]^2}$$

$\downarrow$ $k' = k - px$

$$\Sigma(p) = -ie^2\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+\not{p}x+m_B)\gamma_\mu}{\left(k^2-\Delta+i\varepsilon\right)^2}$$

TERM LINEAR IN k DOES NOT CONTRIBUTE $\downarrow$

$$\Delta \equiv m_B^2(1-x) - p^2x(1-x)$$

$$\underline{\underline{\Delta = (m_B^2 - p^2x)(1-x)}}$$

$$\boxed{\Sigma(p) = -ie^2\int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{p}x+m_B)\gamma_\mu}{\left(k^2-\Delta+i\varepsilon\right)^2}}$$

↳ FOR LARGE k :

INTEGRAND $\sim \frac{d^4k}{k^4}$

$\Downarrow$

INTEGRAL IS UV DIVERGENT

$\downarrow$

WE WILL USE DIMENSIONAL REGULARIZATION TO REGULARIZE THE INTEGRAL

↳ WE WILL DISCUSS 2 SUBTRACTION (RENORMALIZATION) SCHEMES TO SUBTRACT THE UV DIV.

1) (MODIFIED) MINIMAL SUBTRACTION ($\overline{MS}$) => USED IN QCD.

SUBTRACT THE $\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi$ TERMS

2) ON-SHELL RENORMALIZATION SCHEME (OS)

↳ CONVENTIONALLY USED IN QED

CAVEAT : { IN ON-SHELL SCHEME ON WILL INTRODUCE IR DIVERGENCES }

TO AVOID THEM ONE USUALLY INTRODUCES A SMALL PHOTON MASS λ

IN PHYSICAL OBSERVABLES ⇒ ALL IR DIV CANCEL AND PHOTON MASS $\lambda \to 0$.

WHEN WORKING IN ON-SHELL SCHEME ONE THEREFORE REPLACES

$$\Delta \longrightarrow \underline{\underline{\Delta_\gamma = (m_B^2 - p^2 x)(1-x) + \lambda^2 x}}$$

↳ DIMENSIONAL REGULARIZATION

$$\Sigma(p) = -ie^2 \int_0^1 dx\, \mu^{2\varepsilon} \int \frac{d^Dk}{(2\pi)^D}\, \frac{\gamma^\mu(\not p\, x + m_B)\gamma_\mu}{(k^2 - \Delta + i\varepsilon)^2}$$

WITH $\underline{\underline{\Delta_\gamma = (m_B^2 - p^2 x)(1-x)}}$

↓

IN D DIM $\qquad \varepsilon \equiv 2 - D/2$

$$\gamma^\mu \not p\, \gamma_\mu = (2-D)\not p = -2(1-\varepsilon)\not p$$

$$\gamma^\mu \gamma_\mu = D = 4\left(1 - \frac{\varepsilon}{2}\right)$$

$$\Sigma(p) = -2ie^2 \mu^{2\varepsilon} \int_0^1 dx \left[-(1-\varepsilon)\not p\, x + 2\left(1-\frac{\varepsilon}{2}\right) m_B\right]$$

$$\cdot \int \frac{d^Dk}{(2\pi)^D}\, \frac{1}{(k^2 - \Delta + i\varepsilon)^2}$$

$$= 2\, \frac{e^2}{(4\pi)^2} (4\pi\mu^2)^\varepsilon\, \Gamma(\varepsilon)$$

$$\cdot \int_0^1 dx \left[-(1-\varepsilon)\not p\, x + 2\left(1-\frac{\varepsilon}{2}\right) m_B\right] \cdot \left[1 - \varepsilon \ln \Delta\right]$$

$$= \frac{e^2}{8\pi^2} \int_0^1 dx \left\{ \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{\Delta}\right] \cdot \left[-\not p\, x + 2m_B\right] \right.$$

$$\left. + \not p\, x - m_B \right\}$$

$$\boxed{\begin{aligned}\Sigma(p) = \not{p}\cdot\left(-\frac{e^2}{16\pi^2}\right)\Big\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - 1 \\ + 2\int_0^1 dx\; x\; \ln\left(\frac{\mu^2}{\Delta}\right)\Big\} \\ + m_B \cdot \left(\frac{e^2}{4\pi^2}\right)\Big\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi - \frac{1}{2} \\ + \int_0^1 dx\; \ln\left(\frac{\mu^2}{\Delta}\right)\Big\}\end{aligned}}$$

<u>NOTE</u> : TO ORDER e^2 WE CAN REPLACE

$$e^2 \longrightarrow e_R^2 + O(e_R^4)$$

$$e^2 = Z_3^{-1} e_R^2$$

$$= e_R^2 \left(1 + O(e_R^2)\right)$$

$$\frac{e^2}{4\pi} = \frac{e_R^2}{4\pi}\left(1 + O(e_R^2)\right)$$

$$= \alpha_{em}\left(1 + O(\alpha_{em})\right)$$

⟹ WAVE FUNCTION & MASS RENORMALIZATION

↳ IN $\Sigma(p)$ BOTH TERMS IN $\not{p}$ AND IN m_B ARE DIVERGENT : WE NEED TO RENORMALIZE 2 QUANTITIES TO ABSORB DIVERGENCES

- ↳ FERMION WAVE FUNCTION RENORMALIZATION
- ↳ FERMION MASS RENORMALIZATION

↳ WAVE FUNCTION RENORMALIZATION

FERMION PROPAGATOR

$$i\, S_F(x-y) = \langle 0 | T\{ \psi(x) \bar{\psi}(y) \} | 0 \rangle$$

- FIELDS WERE DEFINED IN ABSENCE OF INTERACTIONS
  - ↳ "BARE" FIELDS ψ^B

WHAT WE HAVE CALCULATED IS THE CORRECTION TO BARE PROPAGATOR S^B

$$S_0^B(p) \longrightarrow S^B(p) = \frac{1}{\not{p} - m_B - \Sigma(p)}$$

- WE INTRODUCE PHYSICAL (RENORMALIZED) FIELDS ψ^R THROUGH A RENORMALIZATION CONSTANT Z_2

$$\boxed{\psi^B = Z_2^{1/2}\, \psi^R}$$

- PHYSICAL (RENORMALIZED) PROPAGATOR S^R

$$S^B(p) = Z_2 \, S^R(p)$$

$\downarrow$ $\quad Z_2 \equiv 1 + \delta_2$ $\quad \delta_2$ is $O(e^2)$

↳ IS CALLED A COUNTERTERM

$$\boxed{S^R(p) = \frac{1}{(1+\delta_2)} \, S^B(p)}$$

↳ MASS RENORMALIZATION

WE DEFINE PHYSICAL MASS m^R

THROUGH A MASS RENORMALIZATION CONSTANT Z_m

$$\boxed{m_B = Z_m \, m_R}$$

$Z_m \equiv (1 + \delta_m)$ WITH $\delta_m = O(e^2)$

↳ MASS COUNTERTERM

↳
$$S^R(p) = \frac{1}{(1+\delta_2)\, \not p - (1+\delta_2)(1+\delta_m) m_R - \Sigma(p) + O(e^4)}$$

$$\boxed{S^R(p) = \frac{1}{\not p - m_R + \delta_2 \not p - (\delta_2 + \delta_m) m_R - \Sigma(p) + O(e^4)}}$$

⇒ (MODIFIED) MINIMAL SUBTRACTION SCHEME

↳ MINIMAL SUBTRACTION SCHEME (MS)

WE CHOOSE RENORMALIZATION CONSTANTS SUCH THAT THEY JUST ABSORB THE DIVERGENT $\frac{1}{\varepsilon}$ TERMS IN $\Sigma(p)$

$$\boxed{\delta_2^{MS} \equiv -\frac{\alpha_{em}}{4\pi}\frac{1}{\varepsilon}} \Rightarrow Z_2^{MS} = 1 - \frac{\alpha_{em}}{4\pi}\frac{1}{\varepsilon} + O(\alpha_{em}^2)$$

$$\delta_2^{MS} + \delta_m^{MS} = -\frac{\alpha_{em}}{\pi}\frac{1}{\varepsilon}$$

$$\downarrow$$

$$\boxed{\delta_m^{MS} \equiv -\frac{3\alpha_{em}}{4\pi}\frac{1}{\varepsilon}} \Rightarrow Z_m^{MS} = 1 - \frac{3\alpha_{em}}{4\pi}\frac{1}{\varepsilon} + O(\alpha_{em}^2)$$

↳ MODIFIED MINIMAL SUBTRACTION SCHEME ($\overline{MS}$)

AS $\frac{1}{\varepsilon}$ ALWAYS COMES IN COMBINATION

$$\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)$$

WE CAN DEFINE RENORM. CONSTANTS BY ABSORBING THIS COMBINATION

$$\boxed{\begin{aligned} Z_2^{\overline{MS}} &= 1 - \frac{\alpha_{em}}{4\pi}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right] + O(\alpha_{em}^2) \\ Z_m^{\overline{MS}} &= 1 - \frac{3\alpha_{em}}{4\pi}\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right] + O(\alpha_{em}^2) \end{aligned}}$$

IN THIS SCHEME m_R IS KNOWN AS $\overline{MS}$ MASS

⇒ RENORMALIZED PROPAGATOR / POLE MASS

↳ IN $\overline{MS}$ SCHEME

$$\left[S^R(p)\right]_{\overline{MS}} = \frac{1}{\not p - m_R - \Sigma_R^{\overline{MS}}(p)}$$

$$\Sigma_R^{\overline{MS}}(p) = \not p \left(-\frac{\alpha_{em}}{4\pi}\right)\left\{-1 + 2\int_0^1 dx \; x \; \ln\left(\frac{\mu^2}{\Delta}\right)\right\} + m_R\left(\frac{\alpha_{em}}{\pi}\right)\left\{-\frac{1}{2} + \int_0^1 dx \; \ln\left(\frac{\mu^2}{\Delta}\right)\right\} + O(\alpha_{em}^2)$$

$$\Delta = \left(m_R^2 - p^2 x\right)(1-x)$$

↳ POLE MASS

S^R HAS POLE FOR $\not p = m_P$

↑ POLE MASS (PHYSICAL e^- MASS)

$$\boxed{m_P - m_R = \Sigma_R^{\overline{MS}}(m_P)}$$

$$m_R = m_P - \Sigma_R^{\overline{MS}}(m_P)$$

$$= m_P\left[1 - \frac{\alpha_{em}}{4\pi}\left(-1 + 3\ln\frac{\mu^2}{m_P^2} \underbrace{- 2\int_0^1 dx\,(2-x)\ln\left[(1-x)^2\right]}_{5}\right) + O(\alpha_{em}^2)\right]$$

$\overline{MS}$ MASS

$$\boxed{m_R(\mu) = m_P\left[1 - \frac{\alpha_{em}}{4\pi}\left(4 + 3\ln\frac{\mu^2}{m_P^2}\right) + O(\alpha_{em}^2)\right]}$$

IN $\overline{MS}$ SCHEME

m_R IS FUNCTION OF ARBITRARY SCALE μ

$m_R(\mu)$: RUNNING MASS

FOR $\mu \uparrow \Rightarrow m_R \downarrow$

NOTE : PHYSICAL OBSERVABLES O ARE INDEPENDENT OF μ

$$\frac{dO}{d\mu} = 0$$: RENORMALIZATION GROUP EQ.

↳ RENORMALIZED PROPAGATOR IN $\overline{MS}$ SCHEME

$$[S^R(p)]^{-1} = \not{p}\left[1 + \frac{\alpha_{em}}{4\pi}\left\{-1 + \ln\frac{\mu^2}{m_R^2} - 2\int_0^1 dx\, x \ln\left[\left(1-\frac{p^2}{m_R^2}x\right)(1-x)\right]\right\}\right]$$

$$- m_R\left[1 + \frac{\alpha_{em}}{\pi}\left\{-\frac{1}{2} + \ln\frac{\mu^2}{m_R^2} - \int_0^1 dx\, \ln\left[\left(1-\frac{p^2}{m_R^2}x\right)(1-x)\right]\right\}\right]$$

$$\boxed{\begin{aligned}[S^R(p)]^{-1}_{\overline{MS}} = {} & \not{p}\left[1 + \frac{\alpha_{em}}{4\pi}\left\{\ln\frac{\mu^2}{m_R^2} + \left(1+\frac{m_R^2}{p^2}\right)\left[1 - \left(1-\frac{m_R^2}{p^2}\right)\ln\left(1-\frac{p^2}{m_R^2}\right)\right]\right\}\right] \\ & - m_R\left[1 + \frac{\alpha_{em}}{\pi}\left\{\ln\frac{\mu^2}{m_R^2} + \frac{3}{2} - \left(1-\frac{m_R^2}{p^2}\right)\ln\left(1-\frac{p^2}{m_R^2}\right)\right\}\right] \\ & + O(\alpha_{em}^2)\end{aligned}}$$

⇒ ON-SHELL RENORMALIZATION SCHEME

↳ RENORMALIZED PROPAGATOR

$$S^R(p) = \frac{1}{\not{p} - m_R - \Sigma_R(p)}$$

WITH $\underline{\underline{\Sigma_R(p) \equiv \Sigma(p) - \delta_2 \not{p} + (\delta_2 + \delta_m)\, m_R}}$

↳ FINITE RENORMALIZED SELF-ENERGY

↳ IN QED ON NORMALLY USES ON-SHELL RENORMALIZATION SCHEME

$m_R = m_P$: POLE MASS OF PROPAGATOR (e^- : $m_P = m_e = 0.511$ MeV)

&

RESIDUE OF POLE $\not{p} = m_P$ IN S^R IS UNITY

- $m_R = m_P$ IMPLIES $\boxed{\Sigma_R^{OS}(m_P) = 0}$ → ON-SHELL SCHEME
- RESIDUE OF POLE $\not{p} = m_P$ IS UNITY IMPLIES : $\boxed{\left.\frac{d}{d\not{p}} \Sigma_R^{OS}(p)\right|_{\not{p}=m_P} = 0}$

↳ δ_m^{OS}

$$\Sigma_R^{OS}(m_p) = 0$$

$$\Updownarrow$$

$$\boxed{\Sigma(m_p) = -\delta_m^{OS}\, m_p}$$

$$\delta_m^{OS} = \left(-\frac{\alpha_{em}}{4\pi}\right)\Big\{ 3\left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi\right] - 1 + 3 \ln \frac{\mu^2}{m_p^2} \underbrace{- 2\int_0^1 dx\, (2-x) \ln\left[(1-x)^2 + \cancel{\frac{\lambda^2}{m_p^2}} x\right]}_{5} \Big\}$$

CAN BE NEGLECTED AS $\int$ IS FINITE

$$\boxed{\delta_m^{OS} = \left(-\frac{\alpha_{em}}{4\pi}\right)\left\{ 3\left[\frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_p^2}\right)\right] + 4 \right\}}$$

NOTE : $\delta_m^{OS} - \delta_m^{\overline{MS}} = \left(-\frac{\alpha_{em}}{4\pi}\right)\left\{ 3 \ln \frac{\mu^2}{m_p^2} + 4 \right\}$

FINITE DIFFERENCE

↳ δ_2^{OS}

$$\frac{d}{d\not p}\Sigma_R^{OS}(p)\Big|_{\not p = m_p} = 0$$

$$\Updownarrow$$

$$\boxed{\delta_2^{OS} = \frac{d}{d\not p}\Sigma(p)\Big|_{\not p = m_p}}$$

$$\Sigma(p) = \not p\,\delta_2(p^2) + m_p\,\delta_m(p^2)$$

$$\downarrow \qquad \frac{d}{d\not p}\delta_2(p^2) = \delta_2'(p^2)\,.\,2\not p$$

$$\frac{d}{d\not p}\Sigma(p) = \delta_2(p^2) + \not p\,.\,2\not p\,\delta_2'(p^2) + 2\not p\,m_p\,\delta_m'(p^2)$$

$$\Bigg\| \quad \delta_2^{OS} = \delta_2(m_p^2) + 2m_p^2\left[\delta_2'(m_p^2) + \delta_m'(m_p^2)\right]$$

- $$\delta_2(m_p^2) = -\frac{\alpha_{em}}{4\pi}\left\{\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m_p^2} - 1 \underbrace{- 2\int_0^1 dx\, x\,\ln\left[(1-x)^2 + \cancel{\frac{\lambda^2}{m_p^2}}\,x\right]}_{3}\right\}$$

$$\delta_2(m_p^2) = \left(-\frac{\alpha_{em}}{4\pi}\right)\left\{\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{m_p^2} + 2\right\}$$

- $$\delta_2(p^2) + \delta_m(p^2) = \left(\frac{\alpha_{em}}{4\pi}\right)\left\{ 3\left[\frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_p^2}\right)\right] - 1 - 2\int_0^1 dx\,(2-x)\,\ln\left[\left(1-\frac{p^2}{m_p^2}x\right)(1-x) + \frac{\lambda^2}{m_p^2}x\right]\right\}$$

$$\delta_2'(m_p^2) + \delta_m'(m_p^2) = \left(\frac{\alpha_{em}}{4\pi}\right)\left(\frac{+2}{m_p^2}\right)\int_0^1 dx \quad \frac{(2-x)\,x\,(1-x)}{(1-x)^2 + \frac{\lambda^2}{m_p^2}x}$$

FOR $\lambda^2 = 0$

THIS INTEGRAND $\dot{\sim}$ $\frac{1}{1-x}$

IR DIVERGENT FOR $x \to 1$

AROUND $x = 1$, λ^2 REGULARIZES INTEGRAL

$$\int_0^1 dx \, \frac{\cdots}{(1-x)^2 + \frac{\lambda^2}{m_p^2}x} \simeq \int_0^{1-\lambda/m_p} dx \, \frac{\cdots}{(1-x)^2}$$

+ TERMS $\cancel{O\left(\frac{\lambda^2}{m_p^2}\right)}$

$$\delta_2'(m_p^2) + \delta_m'(m_p^2) = \left(\frac{\alpha_{em}}{4\pi}\right)\frac{2}{m_p^2}\underbrace{\int_0^{1-\lambda/m_p} dx \quad \frac{x\,(2-x)}{(1-x)}}_{-\frac{1}{2}\left(1 + \ln\frac{\lambda^2}{m_p^2} + \cancel{O\left(\frac{\lambda^2}{m_p^2}\right)}\right)}$$

- $$\boxed{\delta_2^{OS} = \left(-\frac{\alpha_{em}}{4\pi}\right)\left\{\frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m_p^2}\right) + 4 + 2\underset{\uparrow}{\ln\frac{\lambda^2}{m_p^2}}\right\}}$$

<u>NOTE</u>: IN ON-SHELL SUBTRACTION SCHEME:

W.F. RENORM. CONSTANT CONTAINS IR DIV

↳ $\left[S^R(p)\right]^{-1}_{OS}$

$$\left[S^R(p)\right]^{-1}_{OS} = \not{p} - m_p$$

$$+ \not{p}\left(\frac{\alpha_{em}}{4\pi}\right)\Big\{-1 - 2\int_0^1 dx\; x \ln\Big[\Big(1-\frac{p^2}{m_p^2}x\Big)(1-x)\Big]$$

$$- 4 - 2\ln\frac{\lambda^2}{m_p^2}\Big\}$$

$$- m_p\left(\frac{\alpha_{em}}{4\pi}\right)\Big\{-2 - 4\int_0^1 dx\; \ln\Big[\Big(1-\frac{p^2}{m_p^2}x\Big)(1-x)\Big]$$

$$- 4 - 2\ln\frac{\lambda^2}{m_p^2} - 4\Big\}$$

$$\boxed{\begin{aligned}\left[S^R(p)\right]^{-1}_{OS} &= (\not{p} - m_p)\left[1 - \left(\frac{\alpha_{em}}{4\pi}\right)2\ln\frac{\lambda^2}{m_p^2}\right] \\ &+ \left(\frac{\alpha_{em}}{4\pi}\right)\Big\{\not{p}\Big[-3 + \frac{m_p^2}{p^2} + \Big(\frac{m_p^2}{p^2}+1\Big)\Big(\frac{m_p^2}{p^2}-1\Big)\ln\Big(1-\frac{p^2}{m_p^2}\Big] \\ &- m_p\Big[-2 + 4\Big(\frac{m_p^2}{p^2}-1\Big)\ln\Big(1-\frac{p^2}{m_p^2}\Big)\Big]\Big\}\end{aligned}}$$

<u>NOTE</u>:
- IR DIV $\sim \ln\frac{\lambda^2}{m_p^2}$ ENTERS DUE TO OUR CHOICE OF ON-SHELL SUBTRACTION SCHEME
- IR DIV WILL CANCEL OUT OF PHYSICAL OBSERVABLES

5) RENORMALIZED PERTURBATION THEORY

⇒ RENORMALIZATION CONSTANTS IN QED

↳ QED LAGRANGIAN : WITH PARAMETERS (UNPHYSICAL)
DEFINED IN ABSENCE OF
INTERACTIONS

$$\mathcal{L}_{QED} = \bar{\psi}_B (i\gamma^\mu \partial_\mu - m_B)\psi_B + e_B \bar{\psi}_B \gamma^\mu \psi_B A_{B\mu}$$

$$- \frac{1}{4} F_{B\mu\nu} F^{B\mu\nu} + \text{GAUGE - FIXING}$$

MULTIPLICATIVE RENORMALIZABILITY :

ALL INFINITIES IN QED CAN BE ABSORBED BY IN RENORMALIZATION CONSTANTS WHICH REDEFINE BARE QUANTITIES INTO PHYSICAL ONES

$\psi_B = \sqrt{Z_2}\,\psi$ ELECTRON FIELD RENORM.

$A^\mu_B = \sqrt{Z_3}\,A^\mu$ PHOTON FIELD RENORM.

$e_B = Z_e\, e_R$ CHARGE RENORM.

$m_B = Z_m\, m_R$ MASS RENORM.

ψ, A^μ, e_R, m_R ARE PHYSICAL QUANTITIES

⟹ COUNTERTERM FEYNMAN RULES

↳ $$\mathcal{L}_{QED} = Z_2 \bar{\psi} (i \gamma^\mu \partial_\mu - Z_m m_R) \psi$$
$$+ Z_e Z_2 \sqrt{Z_3}\ e_R \bar{\psi} \gamma^\mu \psi A_\mu$$
$$- \frac{1}{4} Z_3\ F_{\mu\nu} F^{\mu\nu} + \text{GAUGE-FIXING}$$

↳ COUNTERTERMS δ (START AT ORDER e_R^2)

$$Z_2 = 1 + \delta_2$$
$$Z_m = 1 + \delta_m$$
$$Z_3 = 1 + \delta_3$$
$$Z_1 = 1 + \delta_1$$

$$\underline{\underline{Z_1 \equiv Z_e Z_2 \sqrt{Z_3}}}$$

VERTEX RENORM.

↳ $$\mathcal{L}_{QED} = \bar{\psi} (i \gamma^\mu \partial_\mu - m_R) \psi + e_R \bar{\psi} \gamma^\mu \psi A_\mu$$
$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \text{GAUGE FIXING}$$
$$+ \delta_2 \bar{\psi} i \gamma^\mu \partial_\mu \psi - (\delta_2 + \delta_m) \bar{\psi} m_R \psi$$
$$+ \delta_1 e_R \bar{\psi} \gamma^\mu \psi A_\mu - \delta_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$\mathcal{L}_{QED}$ CONSISTS OF

1) TERMS OF SAME STRUCTURE AS BEFORE BUT NOW WITH PHYSICAL (RENORMALIZED) PARAMETERS

2) COUNTERTERMS WHICH APPEAR AS INTERACTIONS IN $\mathcal{L}$

↳ ADDITIONAL FEYNMAN RULES

- $\delta_2 \bar{\Psi} i \gamma^\mu \partial_\mu \Psi - (\delta_2 + \delta_m) m_R \bar{\Psi} \Psi$

$$\boxed{i \left(\delta_2 \not{p} - (\delta_2 + \delta_m) m_R \right)}$$

- $- \delta_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$i \delta_3 (-g^{\mu\nu} k^2 + k^\mu k^\nu)$$

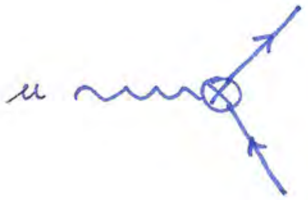
GAUGE FIXING

$$+ \frac{1}{2\xi} i \delta_3 (-k^\mu k^\nu)$$

- FEYNMAN GAUGE ($\xi = 1$)

$$\boxed{i \delta_3 (-g^{\mu\nu}) k^2}$$

- $\delta_1 \, e_R \, \bar{\psi} \gamma^\mu \psi A_\mu$



$$\boxed{i \, \delta_1 \, e_R \, \gamma^\mu}$$

SIGN FOR e^-
$(e_R > 0)$

$\Rightarrow$ 2 - POINT FUNCTION : ELECTRON

- TREE LEVEL

$$\frac{i}{\not{p} - m_R}$$

- 1- LOOP CORR.

$$\frac{i}{\not{p} - m_R} \left(-i \Sigma(\not{p}) \right) \frac{i}{\not{p} - m_R}$$

$$+ \quad \frac{i}{\not{p} - m_R} \, i \left(\delta_2 \not{p} - (\delta_2 + \delta_m) m_R \right) \frac{i}{\not{p} - m_R}$$

- 1- PARTICLE IRREDUCIBLE GRAPHS + COUNTERTERM YIELD

$$\Sigma(\not{p}) - \delta_2 \not{p} + (\delta_2 + \delta_m) m_R$$

$$\equiv \Sigma_R(\not{p})$$

DEFINES FINITE (RENORMALIZED)
SELF ENERGY (SEE P. 68)

- BY SUMMING ALL 1-PARTICLE REDUCIBLE GRAPHS
YIELDS RENORMALIZED (FULL) PROPAGATOR

$$= \frac{i}{\not{p} - m_R - \Sigma_R(\not{p})} \equiv i \, S_R(p)$$

δ_2 & δ_m ABSORB DIVERGENCIES IN $\Sigma(\not{p})$

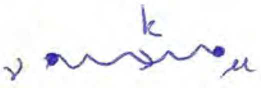
DEFINED IN A PARTICULAR SUBTRACTION SCHEME

WE HAVE DISCUSSED

1) $\overline{MS}$ SCHEME
2) ON-SHELL SCHEME $\boxed{\Sigma_R^{OS}(p=m_R)=0 \;\; \& \;\; \frac{d}{dp}\Sigma_R^{OS}(p=m_R)=0}$

⇒ 2 - POINT FUNCTION : PHOTON

- TREE - LEVEL (FEYNMAN GAUGE)



$$-\frac{i g^{\mu\nu}}{k^2}$$

- 1 - LOOP CORR.

$$-\frac{i g^{\mu\alpha}}{k^2}\left(i\,\Pi_{\alpha\beta}(k)\right)-\frac{i g^{\beta\nu}}{k^2}$$

\+ 

$$-\frac{i g^{\mu\alpha}}{k^2}\left(-i\,\delta_3\, k^2 g_{\alpha\beta}\right)-\frac{i g^{\beta\nu}}{k^2}$$

- 1 - PARTICLE IRREDUCIBLE GRAPHS + COUNTERTERM

$$\Pi_{\alpha\beta}(k)=\left(k^2 g_{\alpha\beta}-k_\alpha k_\beta\right)\Pi(k^2)$$

$\Pi(k^2)-\delta_3 \equiv \Pi_R(k^2)$ DEFINES FINITE (RENORM) PHOTON SELF-ENERGY

δ_3 ABSORBS DIV.

- BY SUMMING ALL 1-PARTICLE REDUCIBLE GRAPHS

  + COUNTERTERMS

  YIELDS RENORMALIZED (FULL) PHOTON PROPAGATOR

$$= \frac{i\left(-g^{\mu\nu}\right)}{k^2\left[1-\Pi_R(k^2)\right]} + k^\mu k^\nu \text{ TERMS}$$

PROPAGATOR HAS POLE AT $k^2=0$ (γ IS MASSLESS)

ENSURED BY
GAUGE INV.

WITH RESIDUE = 1

$$\Downarrow$$

$$\boxed{\Pi_R(0) = 0}$$

$$\Updownarrow$$

$$\boxed{\delta_3 = \Pi(0)}$$

SEE p. 53

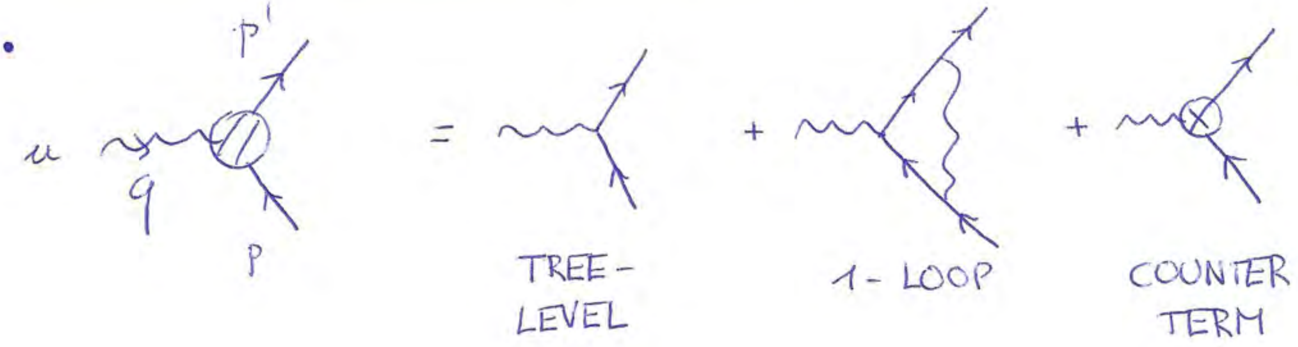
$$\delta_3 = -\frac{e_R^2}{12\pi^2}\left\{\frac{1}{\varepsilon} - \gamma_E - \frac{1}{2}f'(4) + \ln\left(\frac{4\pi\mu^2}{m_R^2}\right)\right\}$$

$$+ O(e_R^4)$$

$$\Pi_R(k^2) = -\frac{e_R^2}{2\pi^2}\int_0^1 dx\; x(1-x)\ \ln\left(\frac{m_R^2}{m_R^2 - k^2 x(1-x)}\right)$$

$$+ O(e_R^4)$$

⟹ 3 - POINT FUNCTION : VERTEX

- 

$$i e_R \Gamma^\mu = i e_R \gamma^\mu + i e_R \left[F_1^{1LOOP}(q^2)\,\gamma^\mu + F_2^{1LOOP}(q^2)\, \frac{i\sigma^{\mu\nu} q_\nu}{2m} \right] + i e_R\, \delta_1\, \gamma^\mu$$

- $F_2^{1\,LOOP}(q^2)$ IS FINITE

WE FOUND $F_2^{1\,LOOP}(0) = \frac{\alpha_{em}}{2\pi}$

- $\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2)$

$$\Downarrow$$

$$F_1(q^2) = \underline{1} + F_1^{1\,LOOP}(q^2) + \delta_1$$

BECAUSE $\boxed{F_1(q^2=0) = \underline{1}}$ CHARGE !

$$\Updownarrow$$

$$\boxed{\delta_1 = - F_1^{1\,LOOP}(0)}$$

$\Longrightarrow$ $Z_1 = Z_2$ PROOF (1 LOOP)

↳ $F_1(q^2)$ TO 1 LOOP

UV DIVERGENT CONTRIBUTION
CAN ONLY COME FROM TERM

$$\mathcal{M}^{\mu,UV}_{(1)}(p',p) = ie\,\bar{u}(p',s') \cdot (-ie^2)\, 2\int_0^1 dx \int_0^x dy\ \mu^{2\varepsilon} \int \frac{d^Dk}{(2\pi)^D} \frac{\gamma^\alpha \not{k} \gamma^\mu \not{k} \gamma_\alpha}{(k^2-\Delta+i\varepsilon)^3}$$

WITH $\Delta \equiv m^2x^2 - q^2y(x-y)$

↓

- $\gamma^\alpha \not{k} \gamma^\mu \not{k} \gamma_\alpha = -2(1-\varepsilon)\not{k}\gamma^\mu\not{k}$

$$= -2(1-\varepsilon)\gamma^\alpha\gamma^\mu\gamma^\beta k_\alpha k_\beta$$

- $$\int \frac{d^Dk}{(2\pi)^D} \frac{k_\alpha k_\beta}{(k^2-\Delta+i\varepsilon)^3} = g_{\alpha\beta} \frac{i\,(4\pi)^\varepsilon}{(4\pi)^2} \frac{\Gamma(\varepsilon)}{2\,\Gamma(3)} \cdot \frac{1}{\Delta^\varepsilon}$$

- $-2(1-\varepsilon)\gamma^\alpha\gamma^\mu\gamma_\alpha = 4\left(1-2\varepsilon+O(\varepsilon^2)\right)\gamma^\mu$

$$F_1^{UV}(0) = \frac{e^2}{(4\pi)^2} (4\pi\mu^2)^{\varepsilon} \frac{\Gamma(\varepsilon)}{4} \, 4(1-2\varepsilon)$$

$$\cdot\, 2 \int_0^1 dx \int_0^x dy \left\{ 1 - \varepsilon \ln m^2 x^2 \right\}$$

$$= \frac{\alpha_{em}}{4\pi} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right) - 2 \right.$$

$$\left. - 4 \underbrace{\int_0^1 dx \; x \ln x}_{-\frac{1}{4}} \right\}$$

$$= \frac{\alpha_{em}}{4\pi} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2}\right) - 1 \right\}$$

$$\Downarrow$$

$$\hookrightarrow \; \delta_1^{\overline{MS}} = - F_1^{UV}(0)$$

$$\delta_1^{\overline{MS}} = - \frac{\alpha_{em}}{4\pi} \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right] + O(\alpha_{em}^2)$$

$$\Downarrow$$

$$\boxed{Z_1^{\overline{MS}} = 1 - \frac{\alpha_{em}}{4\pi} \left[\frac{1}{\varepsilon} - \gamma_E + \ln 4\pi \right] + O(\alpha_{em}^2)}$$

$$\hookrightarrow \quad Z_1^{\overline{MS}} = Z_2^{\overline{MS}}$$

- VALID TO ALL ORDERS IN PERTURBATION THEORY !
(IN QED)

$$\boxed{Z_1 = Z_2} \quad (\text{QED})$$

- IS A CONSEQUENCE OF E.M. GAUGE INVARIANCE

E.M. CURRENT DOES NOT GET RENORMALIZED
$J^\mu = \overline{\psi} \gamma^\mu \psi$ (WARD - TAKAHASHI ID)

- AS A CONSEQUENCE

$$Z_1 = Z_e Z_2 Z_3^{1/2}$$

$$\Downarrow$$

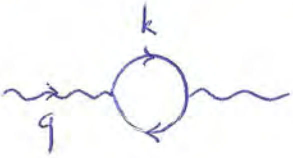
$$\boxed{Z_e = Z_3^{-1/2}}$$

CHARGE RENORMALIZATION
CAN BE OBTAINED FROM CALCULATION
OF VACUUM POL.

6) RENORMALIZABILITY : GENERAL

⇒ SUPERFICIAL DEGREE OF DIVERGENCE (DD).

↳ QED : 1 LOOP

- 

$$\xrightarrow[k>>]{} \int d^4k \, \frac{1}{\not k \not k} \quad : \quad DD = 2$$

DD = # MOMENTA IN NUMERATOR
− # " " DENOMINATOR

SUPERFICIAL DD GIVES MAX. POSSIBLE UV DIV.

FOR VACUUM POL : GAUGE INV.

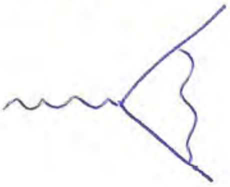
$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi(q^2)$$

$\Pi(q^2)$ HAS 'ONLY' LOGARITHMIC DIV.
↳ RENORM. CONSTANT Z_3

- 
$$\xrightarrow[k>>]{} \int d^4k \, \frac{1}{k^2 \not k} \quad : \quad DD = 1$$

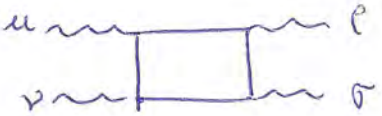
LINEAR DIV.
↳ 2 RENORM. CONSTANTS
Z_2, Z_m

- 3-POINT FUNCTION $\langle \bar{\psi} \psi A \rangle$


$$\longrightarrow \int d^4k \, \frac{1}{\not k \not k \; k^2} \quad : \quad DD = 0$$

LOGARITHMIC DIV.
↳ RENORM. CONSTANT Z_1

- 4-POINT FUNCTION $\langle AAAA \rangle$



$$\longrightarrow \int d^4k \, \frac{1}{\not{k}\not{k}\not{k}\not{k}} \qquad DD = 0.$$

$$M^{\mu\nu\rho\sigma} = \overset{\to \text{DIV.}}{a} \left(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\rho\nu} + g^{\mu\rho} g^{\nu\sigma} \right)$$

$$+ \text{ FINITE TERMS } \sim q_i^\mu$$

BUT GAUGE INV. REQUIRES

$$q_{1\mu} M^{\mu\nu\rho\sigma} = 0 \,, \quad q_{2\nu} M^{\mu\nu\rho\sigma} = 0$$

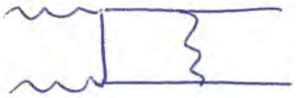
$$q_{3\rho} M^{\mu\nu\rho\sigma} = 0 \,, \quad q_{4\sigma} M^{\mu\nu\rho\sigma} = 0.$$

ONLY POSSIBLE IF $a = 0$

∴ DUE TO GAUGE INV. $\langle A^\mu A^\nu A^\rho A^\sigma \rangle$ IS FINITE

DESPITE $DD = 0$

- 4-POINT FUNCTION $\langle \bar{\psi}\psi AA \rangle$

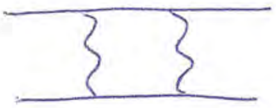


$$\longrightarrow \int d^4k \, \frac{1}{\not{k}\not{k}\not{k}\, k^2} \qquad DD = -1$$

$\downarrow$

FINITE

- 4-POINT FUNCTION $\langle \bar{\psi}\psi\bar{\psi}\psi \rangle$

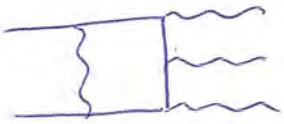


$$\longrightarrow \int d^4k \, \frac{1}{\not{k}\not{k}\, k^2 k^2} \qquad DD = -2$$

$\downarrow$

FINITE

- 5 - POINT FUNCTION $< \bar{\Psi} \Psi AAA >$



$$\longrightarrow \int d^4k \frac{1}{k^2 \not{k} \not{k} \not{k} \not{k}}$$

DD = −2 ↓ FINITE

- ALL n - POINT FUNCTIONS WITH $n \geqslant 5$ HAVE DD < 0 ⟹ FINITE

- AT 1 - LOOP : ONLY $< AA >$
  $< \bar{\Psi} \Psi >$
  $< \bar{\Psi} \Psi A >$
  ARE DIVERGENT ⟹ RENORMALIZATION CONSTANTS Z_3, Z_2, Z_m, Z_1

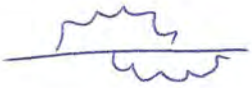
↳ QED : 2 LOOP

- $< AA >$

$$\longrightarrow \int d^8k \frac{1}{\not{k} \not{k} k^2 \not{k} \not{k}}$$

DD = 2

- $< \bar{\Psi} \Psi >$



$$\longrightarrow \int d^8k \frac{1}{\not{k} \not{k} \not{k} k^2 k^2}$$

DD = 1

- $< \bar{\Psi} \Psi A >$

$$\longrightarrow \int d^8k \frac{1}{\not{k} \not{k} \not{k} \not{k} k^2 k^2}$$

DD = 0

- FOR A THEORY LIKE QED

  DD IS SAME FOR 1-LOOP, 2-LOOP, ..., n-LOOP GRAPHS

  ↳ DIVERGENCES CAN BE ABSORBED IN SAME 4 RENORMALIZATION CONSTANTS $Z_3, Z_2, Z_m, \underset{Z_e}{\overset{}{Z_1}}$ ($Z_1 \Updownarrow Z_e$)

e.g.

$\Rightarrow O(\alpha_{em})$ TERM IN Z_3

$\Rightarrow O(\alpha_{em}^2)$ TERM IN Z_3

...

THEORIES FOR WHICH DIV. CAN BE ABSORBED BY A FINITE # OF COUNTERTERMS (4 FOR QED) ARE CALLED RENORMALIZABLE $\Rightarrow$ INFINITE # OF PREDICTIONS !

- FOR QED ($Z_1 = Z_2$)

  WE ONLY NEED TO MEASURE 2 QUANTITIES ! (2 FUNDAMENTAL PARAMETERS)

$\Rightarrow e_R$: RENORM. CHARGE → DETERMINED FROM COULOMB LAW AT LARGE DISTANCE

↓

FIXES $Z_e = Z_3^{-1/2}$ $\qquad \alpha_{em} \equiv \frac{e_R^2}{4\pi} = \frac{1}{137}$

$\Rightarrow m_p$: RENORM. MASS (IN ON-SHELL SCHEME)

∥

POLE MASS $\qquad m_p^{e^-} = 0.511$ MeV

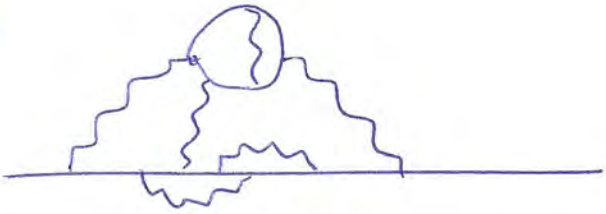
PROPAGATOR HAS POLE AT $\not p = m_p$ → FIXES Z_m

WITH RESIDUE 1 → FIXES Z_2

↳ GENERAL THEORY

HOW TO CALCULATE DD FOR AN ARBITRARY DIAGRAM ?

EX.



- AT EACH VERTEX OF TYPE i
  WE MAY HAVE DERIVATIVE INTERACTIONS
  VERTEX TYPE i ⇒ d_i = # DERIVATIVES IN $\mathcal{L}_{INT}$

→ SPINOR QED $\mathcal{L}_{INT} = e \bar{\Psi} \gamma^\mu \Psi A_\mu$ $(q = -e)$

$d = 0$

→ SCALAR QED

2 TYPES

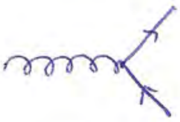
$\mathcal{L}_{INT,1}$

$-iq(p+p')^\mu \Rightarrow d_1 = 1$

$\mathcal{L}_{INT,2}$

$2iq^2 g^{\mu\nu} \Rightarrow d_2 = 0$

→ QCD : HAS 3 TYPES OF INTERACTIONS



$\mathcal{L}_{INT,1}$

$d_1 = 0$

$\mathcal{L}_{INT,2}$

$d_2 = 1$

$\mathcal{L}_{INT,3}$

$d_3 = 0$

- EVERY INTERNAL FERMION LINE

$$\int d^4 p_i \qquad \longrightarrow \qquad \sim \int d^4 p_i \frac{1}{p_i} \qquad DD = 3$$

CONNECTS 2 VERTICES : $\Rightarrow$ ATTRIBUTE $DD = \frac{3}{2}$ TO EACH OF THE 2 VERTICES

$\therefore$ f_i = # FERMION LINES ENTERING VERTEX OF TYPE i

CONTRIBUTE $DD = \frac{3}{2} \cdot f_i$ AT EACH VERTEX

- EVERY INTERNAL BOSON LINE

$$\int d^4 p_i \qquad \rightsquigarrow \qquad \sim \int d^4 p_i \frac{1}{p_i^2} \qquad DD = 2$$

CONNECTS 2 VERTICES $\Rightarrow$ ATTRIBUTE $DD = 1$ TO EACH OF THE 2 VERTICES

$\therefore$ b_i = # BOSON LINES ENTERING VERTEX OF TYPE i

CONTRIBUTE $DD = b_i$ AT EACH VERTEX

- ENERGY - MOMENTUM CONSERVATION

AT EACH VERTEX : THERE IS 1 ENERGY - MOMENTUM CONSERVATION CONDITION

$$\Downarrow$$

$$\delta^4 \left(\sum_i p_i \right)$$

REDUCES DD AT EACH VERTEX BY 4

$\therefore$ CONTRIBUTES $DD = -4$ AT EACH VERTEX

- AT EACH VERTEX OF TYPE i

δ_i = INDEX OF DIVERGENCE OF $\mathcal{L}_{\text{INT},i}$

$$\boxed{\delta_i = d_i + \frac{3}{2} f_i + b_i - 4}$$ → GIVES CONTRIBUTION TO DD AT VERTEX OF TYPE i

→ SPINOR QED

$b = 0$
$f = 2$
$d = 0$
$\Rightarrow \quad \delta = 0$

→ SCALAR QED

$\mathcal{L}_{\text{INT},1}$: $b_1 = 3$
$f_1 = 0$
$d_1 = 1$
$\Rightarrow \quad \delta_1 = 0$

$\mathcal{L}_{\text{INT},2}$: $b_2 = 4$
$f_2 = 0$
$d_2 = 0$
$\Rightarrow \quad \delta_2 = 0.$

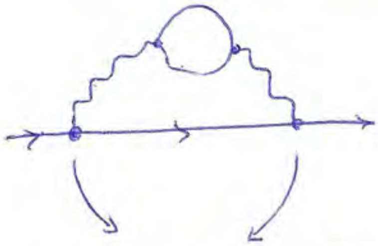
$\therefore$ CONTRIBUTION TO DD WHEN SUMMING OVER ALL VERTICES IN A DIAGRAM

$$DD = \sum_i n_i \, \delta_i$$

$\sum_i$: SUMS OVER ALL TYPES OF INTERACTIONS
SPINOR QED $(i=1)$
SCALAR QED $(i=1,2)$
QCD $(i=1,2,3)$

n_i : # VERTICES OF TYPE i

- CORRECTION FOR EXTERNAL LINES



AT THESE VERTICES WE HAVE EXTERNAL LINES

→ FOR EXTERNAL LINES : NO PROPAGATORS WE SHOULD CORRECT TERMS $\frac{3}{2} f_i$ & b_i FOR EXTERNAL LINES

→ WE SUBTRACTED ONE ENERGY-MOMENTUM TOO MUCH, BECAUSE ONE JUST CORRESPONDS TO OVERALL ENERGY-MOMENTUM CONSERVATION AND DOES NOT LEAD TO A REDUCTION OF DD

- FINAL FORMULA : GENERAL DIAGRAM / GENERAL THEORY

$$\boxed{DD = \sum_i n_i \delta_i + \left(4 - \frac{3}{2}F - B\right)}$$

$$\delta_i = d_i + \frac{3}{2} f_i + b_i - 4$$

F = # EXTERNAL FERMION LINES IN DIAGRAM

B = # EXTERNAL BOSON LINES IN DIAGRAM

⇒ RENORMALIZABLE, NON-RENORMALIZABLE, SUPER-RENORMALIZABLE THEORIES

↳ $\delta_i = 0$: RENORMALIZABLE THEORIES

- IF $\delta_i = d_i + \frac{3}{2} f_i + b_i - 4 = 0$

$$\Downarrow$$

$$\boxed{DD = 4 - \frac{3}{2} F - B}$$

IN RENORMALIZABLE THEORY :
SUPERFICIAL DD ONLY DEPENDS ON EXTERNAL LINES
NOT ON INTERNAL LINES

∴ NO NEW DIVERGENCES APPEAR WHEN CALCULATING HIGHER LOOP DIAGRAMS COMPARED TO 1-LOOP DIAGRAMS

$$\Downarrow$$

WE ONLY NEED TO RENORMALIZE FINITE # PARAMETERS & ALL OTHER QUANTITIES CAN (IN PRINCIPLE) BE PREDICTED

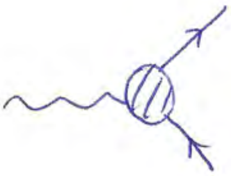
- EXAMPLES ($\delta = 0$) : QED, QCD, STANDARD ELECTROWEAK THEORY

$\equiv$ STANDARD MODEL OF PARTICLE PHYSICS

- QED : ONLY DIVERGENCES

$F = 0, B = 2 \implies DD = 2$

$F = 2,\ B = 0 \Rightarrow DD = 1$



$F = 2,\ B = 1 \Rightarrow DD = 0$

- IN RENORMALIZABLE THEORY : COUPLINGS ARE DIMENSIONLESS !

$$\mathcal{L}_{INT} = -q\,\bar{\psi}\,\gamma^{\mu}\psi\,A_{\mu}$$

$$[\mathcal{L}_{INT}] = 4$$

$$[\psi] = \frac{3}{2}$$

$$[A^{\mu}] = 1$$

$$[q] = 4 - \frac{3}{2}\underset{\substack{\downarrow\\ 2}}{f_i} - \underset{\substack{\downarrow\\ 1}}{b_i} - \underset{\substack{\downarrow\\ 0}}{d_i} \qquad \text{SPINOR QED}$$

$$[q] = -\delta_i \underset{\substack{\uparrow\\ \text{IN RENORMALIZABLE THEORY}}}{=} 0$$

NOTE: INTERACTIONS WITH $\delta = 0$ ARE ALSO CALLED 'MARGINAL' (DIMENSION 0)

↳ $\delta_i > 0$: NON-RENORMALIZABLE THEORIES

- $$DD = \underbrace{\sum_i n_i \delta_i}_{>0} + \left(4 - \frac{3}{2} F - B\right)$$

MORE & MORE DIVERGENCES APPEAR WHEN CALCULATING HIGHER LOOPS

$\Downarrow$

ONE NEEDS AN <u>INFINITE</u> # COUNTERTERMS TO ABSORB ALL DIVERGENCES

- EXAMPLE ($\delta_i > 0$)

$$\mathcal{L}_{INT} = g \, \phi^2 \, \Box \phi^3 \qquad (\phi : \text{SCALAR FIELD})$$

$$\delta_i = \underset{\underset{2}{\shortparallel}}{d_i} + \frac{3}{2} \underset{\underset{0}{\shortparallel}}{f_i} + \underset{\underset{5}{\shortparallel}}{b_i} - 4$$

$$\underline{\underline{\delta_i = 3}}$$

$$[\mathcal{L}] = 4 \quad (\text{ALWAYS})$$

$$[g] = -\delta_i = -3$$

- IN NON-RENORMALIZABLE THEORIES COUPLING'S HAVE NEGATIVE MASS DIMENSIONS

NOTE : COUPLINGS WITH NEGATIVE MASS DIMENSIONS ARE CALLED 'IRRELEVANT'

↳ $\delta_i < 0$: SUPER - RENORMALIZABLE THEORY

- $$DD = \underbrace{\sum_i n_i \cdot \delta_i}_{< 0} + \left(4 - \frac{3}{2} F - B\right)$$

  HIGHER LOOPS BECOME
  MORE CONVERGENT

- EXAMPLE ($\delta_i < 0$)

  $$\mathcal{L}_{INT} = g \phi^3 \qquad (\phi : \text{SCALAR FIELD})$$

  $$\delta_i = \underset{\overset{\shortparallel}{0}}{d_i} + \frac{3}{2} \underset{\overset{\shortparallel}{0}}{f_i} + \underset{\overset{\shortparallel}{3}}{b_i} - 4$$

  $$\underline{\underline{\delta_i = -1}}$$

  $$[g] = -\delta_i = +1$$

- IN SUPER - RENORMALIZABLE THEORIES COUPLINGS HAVE POSITIVE MASS DIMENSIONS

  NOTE : COUPLINGS WITH POSITIVE MASS DIMENSIONS ARE CALLED 'RELEVANT'

⟹ NON-RENORMALIZABLE THEORIES AS EFFECTIVE FIELD THEORIES

↳ EXAMPLE : SCALAR FIELD THEORY WITH DERIVATIVE INTERACTIONS

- $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \underbrace{\frac{g}{4!}\phi^2 \Box \phi^2}_{\mathcal{L}_{INT}}$$

$$\mathcal{L}_{INT} \Rightarrow \delta = d + \frac{3}{2} f + b - 4$$

$$= 2 + 0 + 4 - 4 = 2 \Rightarrow \text{NON-RENORMALIZABLE THEORY}$$

$$[g] = -\delta = -2.$$

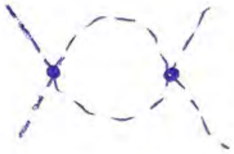
- 4-POINT FUNCTION

→ TREE-LEVEL

$$\mathcal{M}^{(0)} \sim g\, p^2$$

WHERE p STANDS FOR A GENERIC EXTERNAL MOMENTUM

→ 1-LOOP



$$DD = 2 \cdot \underset{\underset{2}{\shortparallel}}{\delta} + 4 - \underset{\underset{4}{\shortparallel}}{B} = 4$$

USE UV REGULATOR Λ (e.g. PAULI-VILLARS MASS SCALE)

$$\mathcal{M}^{(1)} = g^2 \left(c_1 \Lambda^4 + c_2 \Lambda^2 p^2 + c_3 p^4 \ln \Lambda + \cdots \right)$$

$C_2 \Lambda^2 p^2$ DIVERGENCE CAN BE REMOVED BY RENORMALIZING g AS ITS TREE-LEVEL CONTRIBUTION IS $\sim p^2$

BUT $C_1 \Lambda^4$ & $C_3 p^4 \ln \Lambda$ TERMS AT 1-LOOP REQUIRE NEW TREE LEVEL TERMS IN ORDER TO ABSORB THESE DIVERGENCES

- CONSIDER THE EXTENDED $\mathcal{L}$

$$\boxed{\begin{aligned}\mathcal{L} = &\tfrac{1}{2}(\partial_\mu \emptyset)(\partial^\mu \emptyset) - \tfrac{1}{2} m^2 \emptyset^2 \\ &+ \lambda_R Z_\lambda \emptyset^4 + g_R Z_g \emptyset^2 \Box \emptyset^2 + K_R Z_K \emptyset^2 \Box^2 \emptyset^2 + \cdots\end{aligned}}$$

$$\downarrow \quad Z_\lambda = 1 + \delta_\lambda \ , \quad Z_g = 1 + \delta_g \ , \quad Z_K = 1 + \delta_K$$

$$= \tfrac{1}{2}(\partial_\mu \emptyset)(\partial^\mu \emptyset) - \tfrac{1}{2} m^2 \emptyset^2$$
$$+ \lambda_R \emptyset^4 + g_R \emptyset^2 \Box \emptyset^2 + K_R \emptyset^2 \Box^2 \emptyset^2 + \cdots$$
$$+ \lambda_R \delta_\lambda \emptyset^4 + g_R \delta_g \emptyset^2 \Box \emptyset^2 + K_R \delta_K \emptyset^2 \Box^2 \emptyset^2 + \cdots$$

$\downarrow$ LAST LINE CORRESPONDS WITH TER TERMS

$$\sim \lambda_R \delta_\lambda + g_R \delta_g p^2 + K_R \delta_K p^4$$

(POLYNOMIAL IN EXTERNAL MOMENTA)

$\lambda_R \delta_\lambda$ WILL ABSORB $C_1 \Lambda^4$ DIV.

$g_R \delta_g p^2$ WILL ABSORB $C_2 \Lambda^2 p^2$ DIV.

$K_R \delta_K p^4$ WILL ABSORB $C_3 p^4 \ln \Lambda$ DIV.

NOTE : ALSO K_R TERM WILL GENERATE NEW INFINITIES WHEN USED IN LOOP

↳ CAN ALSO BE REMOVED WHEN ADDING ALL POSSIBLE TERMS TO $\mathcal{L}$, CONSISTENT WITH SYMMETRIES

∴ THIS WILL BE POSSIBLE PROVIDED THAT DIVERGENCES MULTIPLY FUNCTIONS THAT ARE POLYNOMIALS IN EXTERNAL MOMENTA

- THEOREM (WEINBERG , 1995)

DIVERGENCES FROM LOOP INTEGRALS ARE POLYNOMIAL TERMS IN EXTERNAL MOMENTA

↳ CAN BE ABSORBED BY TREE - LEVEL TERMS

PROOF :

GENERAL DIVERGENT INTEGRAL IS OF FORM

$$I(p) = \int_0^\infty dk \, \frac{k}{(k+p)}$$

MUST HAVE AT LEAST 1 DENOMINATOR WITH p IN IT (OTHERWISE IT WILL YIELD A RESULT WHICH MULTIPLIES A POLYNOMIAL IN p)

IF ONE DIFFERENTIATES ENOUGH w.r.t p

THEN INTEGRAL BECOMES CONVERGENT

IN THIS CASE : ONE NEED TO DIFF. TWICE :

$$I''(p) = \int_0^\infty dk \, \frac{2k}{(k+p)^3} = \frac{1}{p} \underbrace{\int_0^\infty d\tilde{k} \, \frac{2\tilde{k}}{(1+\tilde{k})^3}}_{1}$$

$$I''(p) = \frac{1}{p}$$

$\downarrow$ INTEGRATE

$$I'(p) = \ln p - \boxed{\ln \Lambda}$$

↳ INTEGRATION CONSTANT

$\downarrow$ INTEGRATE

$$I(p) = p \ln p - p - p \ln \Lambda + \boxed{C_1 \Lambda}$$

↳ INTEGRATION CONSTANT

$$I(p) = p \ln p - p(1 + \ln \Lambda) + C_1 \Lambda$$

↑

FINITE TERM
IS
NON - ANALYTIC IN p :
PREDICTION OF
NON - RENORM. THEORY !

DIVERGENT TERMS
ARE POLYNOMIALS IN p :
CAN BE ABSORBED
BY TREE - LEVEL TERMS IN $\mathcal{L}$

∴ THE NON - ANALYTIC TERMS IN EXTERNAL MOMENTA (e.g. DUE TO BRANCH CUTS , POLES) ARE FULLY PREDICTED BY NON - RENORMALIZABLE THEORIES! $\Rightarrow$ QUANTUM PREDICTIONS

- PREDICTIVE POWER

MAKE COUPLINGS OF NON-RENORM. TERMS IN $\mathcal{L}$ DIMENSIONLESS BY INTRODUCING A MASS SCALE M

$$\mathcal{L} = +\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \lambda \phi^4 + \frac{g_1}{M^2}\phi^2 \Box \phi^2 + \frac{g_2}{M^4}\phi^2 \Box^2 \phi^2 + \cdots$$

WHEN CALCULATING 4-POINT FUNCTION AS FUNCTION OF ENERGY E (e.g. C.M. ENERGY) WE OBTAIN

$$\langle \phi^4 \rangle = \lambda_R + g_{1R}\frac{E^2}{M^2} + g_{2R}\frac{E^4}{M^4} + \cdots$$

+ NON-ANALYTIC TERMS IN TERMS OF RENORM. COUPLINGS

FOR $E << M$

ONLY A FINITE # TERMS ARE IMPORTANT IN EXPANSION

$\Downarrow$

IF ONE CAN FIX THEM BY A FINITE # MEASUREMENTS

$\Downarrow$

! THEORY IS FULLY PREDICTIVE FOR $E << M$

(EFFECTIVE FIELD THEORY)