III. ABELIAN GAUGE THEORY AND ITS QUANTIZATION

1) ABELIAN GAUGE THEORY, QED

2) QUANTIZATION OF PHOTON FIELD

3) FEYNMAN PROPAGATOR FOR PHOTONS

1) ABELIAN GAUGE THEORY (QED)

⇒ LOCAL PHASE TRANSFORMATION

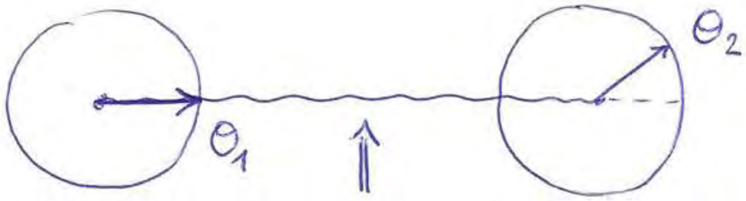
↳ FERMION (SPIN 1/2)

$$\mathcal{L}_{DIRAC} = \bar{\psi}(x)\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x)$$

↳ LOCAL PHASE TF. : U(1) GROUP

$$\psi(x) \xrightarrow{U(1)} e^{i\Theta(x)}\psi(x)$$

$\Theta(x)$ → DEPENDS ON SPACE-TIME POINT



$$\Delta\Theta = \Theta_2 - \Theta_1$$

SIGNAL WHICH COMMUNICATES PHASE DIFFERENCE ("PHOTON")

↳ TRANSFORMATION OF $\mathcal{L}_{DIRAC}$ UNDER LOCAL PHASE TF.

$$\partial_{\mu}\psi(x) \longrightarrow e^{i\Theta(x)}\left[\partial_{\mu}\psi(x) + i(\partial_{\mu}\Theta)\psi\right]$$

$$\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi \longrightarrow \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi + i(\partial_{\mu}\Theta)\bar{\psi}\gamma^{\mu}\psi$$

$$\bar{\psi}\psi \longrightarrow \bar{\psi}\psi$$

$$\mathcal{L}_{DIRAC} \longrightarrow \mathcal{L}_{DIRAC} - (\partial_{\mu}\Theta)\overbrace{\bar{\psi}\gamma^{\mu}\psi}^{\text{VECTOR}}$$

↳ INVARIANCE UNDER LOCAL PHASE TF.

- $$\mathcal{L}_{DIRAC} \xrightarrow{U(1)} \mathcal{L}_{DIRAC} \underbrace{- (\partial_\mu \theta)\, \bar{\psi} \gamma^\mu \psi}_{\text{EXTRA TERM}}$$

  EXTRA TERM

  $\sim (\partial_\mu \theta)$

  DIFFERENCE IN PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- TO MAKE $\mathcal{L}$ INVARIANT UNDER LOCAL PHASE TF $\Rightarrow$ INTRODUCE VECTOR FIELD (GAUGE FIELD) WHICH COMPENSATES FOR DIFFERENT CHOICES OF PHASE BETWEEN DIFFERENT SPACE-TIME POINTS

- INTRODUCE COVARIANT DERIVATIVE

$$\partial_\mu \underset{\text{REPLACE}}{\Longrightarrow} \underline{\underline{D_\mu = \partial_\mu + i e A_\mu}}$$

VECTOR FIELD

ELECTRIC CHARGE
COUPLING OF VECTOR FIELD TO DIRAC FIELD

$$A^\mu \xrightarrow{U(1)} A'^\mu$$

CHOOSE A'^μ SUCH THAT TOTAL $\mathcal{L}$ IS INVARIANT UNDER LOCAL PHASE TF.

- $D_\mu \psi = \partial_\mu \psi + ie A_\mu \psi$

$$\xrightarrow{U(1)} e^{i\theta(x)} \left[\partial_\mu \psi + i(\partial_\mu \theta)\psi + ie A'_\mu \psi \right]$$

$$= e^{i\theta(x)} \left[\partial_\mu \psi + ie\left(A'_\mu + \frac{1}{e}\partial_\mu \theta\right)\psi \right]$$

CHOOSE $A'_\mu = A_\mu - \frac{1}{e}\partial_\mu \theta$

$$\Downarrow$$

$$D_\mu \psi \xrightarrow{U(1)} e^{i\theta(x)} \left[\partial_\mu \psi + ie A_\mu \psi \right]$$

$$\underline{\underline{D_\mu \psi \xrightarrow{U(1)} e^{i\theta(x)} D_\mu \psi}}$$

$$\bar{\psi} i \gamma^\mu D_\mu \psi \xrightarrow{U(1)} \bar{\psi} i \gamma^\mu D_\mu \psi$$

INVARIANT UNDER LOCAL U(1) TF !

$\therefore$

$$\boxed{\begin{array}{l} \text{LAGRANGIAN } \mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \\ \text{IS INVARIANT UNDER LOCAL PHASE TF :} \\ \quad \psi(x) \xrightarrow{U(1)} e^{i\theta(x)} \psi(x) \\ \quad A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu \theta \end{array}}$$

⟹ GEOMETRIC INTERPRETATION OF COVARIANT DERIVATIVE

↳ NORMAL DERIVATIVE

dx^μ : INFINITESIMAL DISTANCE BETWEEN 2 SPACE-TIME POINTS

$$x^\mu \longrightarrow x^\mu + dx^\mu$$

$$\psi(x+dx) - \psi(x) = dx^\mu \, \partial_\mu \psi$$

↳ COVARIANT DERIVATIVE

$$dx^\mu D_\mu \psi = dx^\mu \left(\partial_\mu \psi + ie A_\mu \psi \right)$$

↓ CHOOSE $A_\mu = -\frac{1}{e} \partial_\mu \theta$

(PURE GAUGE FIELD)

$$= dx^\mu \left(\partial_\mu \psi - i (\partial_\mu \theta) \psi \right)$$

$$= \psi(x+dx) - \psi(x) - i \left[\theta(x+dx) - \theta(x) \right] \psi(x)$$

$$= \psi(x+dx) - \left[1 + i \left(\theta(x+dx) - \theta(x) \right) \right] \psi(x)$$

$$\underset{\substack{\Uparrow \\ \text{INFINITESIMAL}}}{\simeq} \psi(x+dx) - \exp\left\{ i \left[\theta(x+dx) - \theta(x) \right] \right\} \psi(x)$$

↳ PARALLEL TRANSPORT

- $\underline{\Phi}(x+dx, x) = \exp\left\{ i \left[\theta(x+dx) - \theta(x) \right] \right\}$

↳ FACTOR WHICH COMPENSATES FOR PHASE DIFFERENCE BETWEEN 2 NEIGHBOURING POINTS.

$$\underline{\Phi}(x+dx, x) = \exp\left\{ i \, dx_\mu \, \partial^\mu \theta \right\}$$

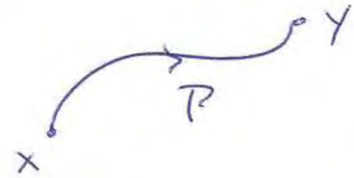
↳ PARALLEL TRANSPORT

$$\underline{\Phi}(x+dx, x) = \exp\left\{ -ie \, dx_\mu \, A^\mu \right\}$$

$$dx_\mu \, D^\mu \psi(x) = \psi(x+dx) - \underline{\Phi}(x+dx, x)\, \psi(x).$$

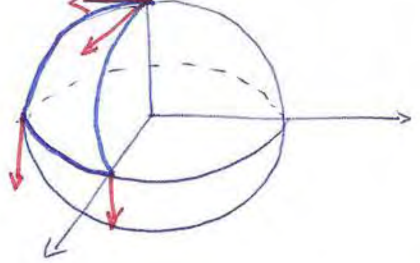
∴ GAUGE FIELD IS EQUIVALENT TO A PHASE DIFFERENCE FOR ψ

- FOR FINITE PATH



$$\underset{P}{\underline{\Phi}}(y, x) = \exp\left\{ -ie \int_P dx_\mu \, A^\mu \right\}.$$

↳ GEOMETRIC ANALOGY (PERSON LIVING ON SURFACE OF SPHERE GOING AROUND CLOSED CURVE WITH ARROW)



$\Delta\theta = 90°$

90° PHASE DIFFERENCE !

⟹ INTERACTION BETWEEN MATTER & GAUGE FIELD

↳ SYMMERY OF $\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ UNDER LOCAL $U(1)$ IS CALLED ABELIAN GAUGE SYMMETRY

↓

REFERS TO GAUGE GROUP $U(1)$

↳ COUPLING TO GAUGE (PHOTON) FIELD

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

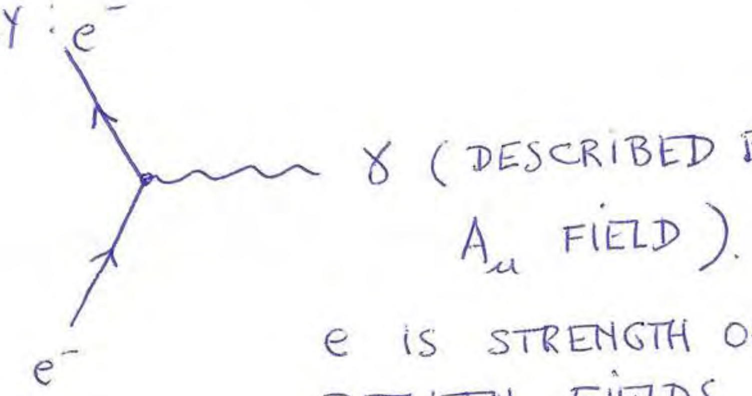
$$= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e(\bar{\psi}\gamma^\mu\psi)A_\mu.$$

$$= \mathcal{L}_{DIRAC} + \mathcal{L}_{INT}$$

↑ INTERACTION LAGRANGIAN

$$\boxed{\mathcal{L}_{INT} = -e(\bar{\psi}\gamma^\mu\psi)A_\mu.}$$

GRAPHICALLY:



e IS STRENGTH OF COUPLING BETWEEN FIELDS

$\Rightarrow$ QUANTUM ELECTRODYNAMICS (QED)

$\hookrightarrow$ INTERACTION BETWEEN MATTER (SPIN 1/2) AND GAUGE FIELDS (PHOTONS) IS DESCRIBED BY

$$\boxed{\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

- FREE MATTER FIELD ψ

$$\mathcal{L}_{DIRAC} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

- INTERACTION BETWEEN MATTER & GAUGE FIELDS

$$\mathcal{L}_{INT} = -e\bar{\psi}\gamma^\mu\psi A_\mu$$

$$\equiv -\underbrace{J^\mu_{em}(x)}_{\text{ELECTROMAGNETIC CURRENT}} A_\mu$$

$$J^\mu_{em} = e\bar{\psi}\gamma^\mu\psi$$

- FREE PHOTON FIELD

FIELD TENSOR $\underline{\underline{F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu}}$.

UNDER $U(1)$ $\quad A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e}\partial^\mu\theta$

$$F^{\mu\nu} \xrightarrow{U(1)} F^{\mu\nu}$$

$$\mathcal{L}_{em} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

↳ $\mathcal{L}_{QED}$ IS INVARIANT UNDER LOCAL $U(1)$!

$$\psi(x) \xrightarrow{U(1)} e^{i\Theta(x)} \psi(x)$$

$$A^\mu(x) \xrightarrow{U(1)} A^\mu(x) - \frac{1}{e} \partial^\mu \Theta$$

$U(1)$ GAUGE SYMMETRY

↳ FIELD EQUATIONS FOR A^μ

$$\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} = 0$$

$\downarrow$ FOR $\phi_r = A_\nu$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = - J^\nu_{em}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2} \cdot 2 \cdot F^{\mu\nu}$$

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu_{em}}$$

INHOMOGENEOUS MAXWELL EQ !

↳ EM CURRENT : SOURCE TERM IN MAXWELL EQ.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = J^\nu_{em}$$

↳ GAUGE INVARIANCE

$$A^\mu \xrightarrow{U(1)} A^\mu - \frac{1}{e} \partial^\mu \theta$$

FREEDOM TO CONSTRAIN A^μ

POSSIBLE CHOICES OF GAUGE ARE

- $\partial_\mu A^\mu = 0$: LORENZ GAUGE (COVARIANT)
- $\overline{\nabla}.\overline{A} = 0$ $(A^0 = 0)$: COULOMB GAUGE (NON - COVARIANT)
  ↳ USED FOR FREE FIELDS
  ($J^\nu_{em} = 0$, ABSENCE OF SOURCES)
- $n_\mu A^\mu = 0$: AXIAL GAUGE
  WITH $n_\mu n^\mu = -1$
  e.g. $n^\mu (0, 0, 0, 1)$

IN FOLLOWING WE WILL OFTEN USE
LORENZ GAUGE $\partial_\mu A^\mu = 0$

↓

$$\Box A^\nu = J^\nu_{em}$$

$$\partial_\mu A^\mu = 0 \qquad \text{CONSTRAINT}$$

GAUGE INVARIANCE ENSURES THAT RESULTS FOR PHYSICAL OBSERVABLES ARE INVARIANT UNDER CHOICE OF GAUGE

2) QUANTIZATION OF PHOTON FIELD

⇒ PRESENCE OF CONSTRAINTS : ISSUES

↳ $$\Box A^{\nu} = J^{\nu}_{em}$$

$$\partial_{\nu} A^{\nu} = 0 \qquad \text{(LORENZ GAUGE)}$$

HOW TO QUANTISE A^{ν} IN PRESENCE OF CONSTRAINT $\partial_{\nu} A^{\nu} = 0$.

↳ ISSUE 1 : CANONICAL MOMENTA

$$A^{\nu} \longrightarrow \Pi^{\nu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\nu}} = - F^{0\nu}$$

$$\Pi^{0} = 0$$

$$\Pi^{i} = - F^{0i} = - \partial^{0} A^{i} + \partial^{i} A^{0}$$

$$= \left(- \bar{\nabla} A^{0} - \frac{\partial \bar{A}}{\partial t} \right)^{i}$$

$$= E^{i} \quad \text{(ELECTRIC FIELD)}$$

FOR A^1, A^2, A^3 OK

FOR A^0 ⇒ WE CANNOT IMPOSE COMMUTATION RELATIONS BECAUSE $\Pi^0 = 0$

↳ ISSUE 2 PHOTON PROPAGATOR

- KLEIN-GORDON (SPIN-0)

$$(\Box + m^2)\,\phi(x) = 0$$

$\downarrow$ FOR SOLUTION $\phi(x) \sim e^{-ik.x}$

$$(-k^2 + m^2)\,\phi = 0$$

SCALAR PROPAGATOR $\dfrac{i}{k^2 - m^2 + i\varepsilon}$

IS INVERSE OF OPERATOR IN FREE FIELD EQUATION (GREEN'S FUNCTION)

$$(\Box_x + m^2)\,\Delta_F(x) = -\delta^4(x)$$

- FREE FIELD EQUATION FOR A^ν

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

$\downarrow$ FOR SOLUTION $\sim e^{-ik.x}$

$$\underbrace{(-k^2 g^{\mu\nu} + k^\mu k^\nu)}\,A_\nu = 0$$

WE LOOK FOR INVERSE OF THIS OPERATOR

$$\left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right)\left(A\, g_{\nu\lambda} + B\, k_\nu k_\lambda\right) = g^\mu_\lambda$$

$$-A k^2 g^\mu_\lambda + A\, k^\mu k_\lambda - \cancel{k^2 B\, k^\mu k_\lambda}$$
$$+ \cancel{k^2 B\, k^\mu k_\lambda} = g^\mu_\lambda$$

$$\begin{cases} -A k^2 = 1 \\ A = 0 \end{cases}$$

$\hookrightarrow$ IMPOSSIBLE FOR ANY CHOICE OF A

$\therefore$ OPERATOR IN FIELD EQUATION FOR A^ν HAS NO INVERSE

⇒ GAUGE FIXING

↳ TO SOLVE ABOVE PROBLEMS WHEN QUANTIZING THE ABELIAN GAUGE THEORY CONSIDER ALTERNATIVE $\mathcal{L}$

$$\mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$ DESCRIBES FREE PHOTON FIELD

$$\mathcal{L}_{em} \longrightarrow \boxed{\mathcal{L}'_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2}$$

$\mathcal{L}_{em}$ + $\mathcal{L}_{GF}$ (GAUGE FIXING)

↳ EQUATIONS OF MOTION

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = - F^{\mu\nu} - (\partial_\alpha A^\alpha)\, g^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu(\partial_\alpha A^\alpha) = 0$$

$$\Box A^\nu - \cancel{\partial^\nu(\partial_\mu A^\mu)} + \cancel{\partial^\nu(\partial_\alpha A^\alpha)} = 0$$

$$\Box A^\nu = 0$$ FIELD EQ. CORRESPONDING WITH $\mathcal{L}'_{em}$ CORRESPOND WITH CHOICE OF LORENZ GAUGE

↳ CANONICAL MOMENTA

$$A^\nu \to \Pi^\nu = -F^{0\nu} - (\partial_\alpha A^\alpha)\, g^{0\nu}$$

$$\begin{cases} \Pi^0 = -\partial_\alpha A^\alpha \\ \Pi^i = -F^{0i} \end{cases}$$

, OK ISSUE 1 IF $\partial_\alpha A^\alpha$ CONSIDERED AS OPERATOR

↳ EQUAL TIME COMMUTATION RELATIONS (ETCR)

TO QUANTIZE THE THEORY, WE IMPOSE

$$\left[A^\mu(\bar{x},t),\, A^\nu(\bar{x}',t)\right]_- = 0$$

$$\left[\Pi^\mu(\bar{x},t),\, \Pi^\nu(\bar{x}',t)\right]_- = 0$$

$$\left[A^\mu(\bar{x},t),\, \Pi^\nu(\bar{x}',t)\right]_- = i\, g^{\mu\nu}\, \delta^3(\bar{x}-\bar{x}')$$

WITH $[A,B]_- = AB - BA$ COMMUTATOR

SPIN 1 FIELD DESCRIBES BOSONS (PHOTONS)

A^μ, Π^μ CONSIDERED AS OPERATORS

$$\rightsquigarrow \left[A^\mu(\bar{x},t),\, \Pi^0(\bar{x}',t)\right]_- = i\, g^{\mu 0}\, \delta^3(\bar{x}-\bar{x}')$$

$$= -\left[A^\mu(\bar{x},t),\, \dot{A}^0(\bar{x}',t)\right]_- - \bar{\nabla}_{x'} \cdot \underbrace{\cancel{\left[A^\mu(\bar{x},t),\, \bar{A}(\bar{x}',t)\right]_-}}_{0}$$

$$\rightsquigarrow \left[A^{\mu}(\bar{x},t) , \underbrace{\Pi^{i}(\bar{x}',t)}_{-\dot{A}^{i} + \partial^{i} A^{0}} \right]_- = i g^{\mu i} \delta^3(\bar{x}-\bar{x}')$$

$$= -\left[A^{\mu}(\bar{x},t), \dot{A}^{i}(\bar{x}',t) \right]_- - \bar{\nabla}^{i}_{x'} \cdot \underbrace{\cancel{\left[A^{\mu}(\bar{x},t), A^{0}(\bar{x}',t) \right]_-}}_{0}$$

$$\therefore \quad \left[A^{\mu}(\bar{x},t), \dot{A}^{\nu}(\bar{x}',t) \right]_- = -i g^{\mu\nu} \delta^3(\bar{x}-\bar{x}')$$

$$\hookrightarrow \mu = 0 : \quad \boxed{\left[A^{\mu}(\bar{x},t), \dot{A}^{0}(\bar{x}',t) \right]_- = -i \, \delta^3(\bar{x}-\bar{x}')}$$

$$\hookrightarrow \mu = i : \quad \boxed{\left[A^{\mu}(\bar{x},t), \dot{A}^{i}(\bar{x}',t) \right]_- = +i \, \delta^3(\bar{x}-\bar{x}')}$$

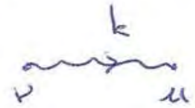
↳ PHOTON PROPAGATOR (IN LORENZ GAUGE)

$$\Box A^{\mu} = 0$$

$$\underbrace{\left(-k^2 g^{\mu\nu} \right)}_{\text{INVERSE EXISTS}} A_{\nu} = 0$$

$$\left(-k^2 g^{\mu\nu} \right) \cdot \left(A \, g_{\nu\lambda} + B \, k_{\nu} k_{\lambda} \right) = g^{\mu}{}_{\lambda}$$

$$A = -\frac{1}{k^2}, \qquad B = 0 \quad \Longrightarrow \quad \text{PHOTON PROPAGATOR} \sim \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}$$



$\Rightarrow$ NORMAL MODE EXPANSION OF $A^\mu(\bar{x},t)$

↳ UPON QUANTIZATION

CLASSICAL FIELDS A^μ

ARE RE-INTERPRETED AS FIELD OPERATORS $\hat{A}^\mu$

THAT SATISFY ETCR

(NOTE: FOR SIMPLICITY WE WILL DROP $\wedge$ NOTATION IN QFT THE FIELDS ARE UNDERSTOOD AS OPERATORS)

↳ POLARIZATION VECTORS:

A NORMAL MODE SOLUTION IS CHARACTERIZED

BY $\sim e^{ik.x}\,\mathcal{E}^\mu(\bar{k},\lambda) \qquad \lambda = 0,1,2,3$

WITH $k(\omega_{\bar{k}}, \underbrace{0,0,|\bar{k}|}_{\bar{k}})$ CONVENIENT CHOICE $\bar{k} = |\bar{k}|\,\bar{e}_z$

$\underline{\underline{\omega_{\bar{k}} = |\bar{k}|}}$

$\mathcal{E}^\mu(\bar{k},\lambda=0) = (1,0,0,0)$ SCALAR POL.

$$\left.\begin{aligned}\mathcal{E}^\mu(\bar{k},\lambda=1) &= (0,1,0,0)\\ \mathcal{E}^\mu(\bar{k},\lambda=2) &= (0,0,1,0)\end{aligned}\right\}\ \text{TRANSVERSE POL.}\quad 0 = \bar{k}\cdot\bar{\mathcal{E}}(\bar{k},\lambda=\tfrac{1}{2})$$

$\mathcal{E}^\mu(\bar{k},\lambda=3) = (0,0,0,1)$ LONGITUDINAL POL.

NORMALIZATION $\quad \mathcal{E}^\mu(\bar{k},\lambda)\,\mathcal{E}^*_\mu(\bar{k},\lambda') = -\zeta_\lambda\,\delta_{\lambda\lambda'}$

$(\zeta_0 \equiv -1,\ \zeta_i \equiv +1)$

↳ NORMAL MODE EXPANSION

$$\boxed{A^\mu(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} \left\{ a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\,e^{-ik.x} + a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\,e^{+ik.x} \right\}}$$

$\omega_{\bar{k}} = |\bar{k}|$ PARTICLE WITH MASS 0 (PHOTON)
(→ MOVES WITH SPEED OF LIGHT)

$a(\bar{k},\lambda)$ ANNIHILATES PHOTON WITH MOMENTUM $\bar{k}$ & POLARIZATION λ

$a^+(\bar{k},\lambda)$ CREATES PHOTON " "

⟹ COMMUTATION RELATIONS FOR a, a^+

↳ $\int d^3\bar{x}\; e^{-i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) + \frac{i}{\omega_{\bar{k}}}\dot{A}^\mu(\bar{x},t) \right\}$

$$= 2\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a(\bar{k},\lambda)\,\varepsilon^\mu(\bar{k},\lambda)\; e^{-i\omega_{\bar{k}}t}$$

↳ $\int d^3\bar{x}\; e^{+i\bar{k}.\bar{x}} \left\{ A^\mu(\bar{x},t) - \frac{i}{\omega_{\bar{k}}}\dot{A}^\mu(\bar{x},t) \right\}$

$$= 2\,\frac{1}{\sqrt{2\omega_{\bar{k}}}} \sum_{\lambda=0}^{3} a^+(\bar{k},\lambda)\,\varepsilon^{\mu *}(\bar{k},\lambda)\; e^{i\omega_{\bar{k}}t}$$

$\downarrow$ USING $\varepsilon^{\mu}(\bar{k},\lambda)\, \varepsilon^{*}_{\mu}(\bar{k},\lambda') = -\zeta_{\lambda}\, \delta_{\lambda\lambda'}$

$$a(\bar{k},\lambda) = -\zeta_{\lambda} \tfrac{1}{2} \sqrt{2\omega_{\bar{k}}}\, e^{+i\omega_{\bar{k}} t} \int d^3\bar{x}\, e^{-i\bar{k}.\bar{x}}\, \varepsilon^{*}_{\mu}(\bar{k},\lambda) \left\{ A^{\mu} + \frac{i}{\omega_{\bar{k}}} \dot{A}^{\mu} \right\}$$

$$a^{+}(\bar{k},\lambda) = -\zeta_{\lambda} \tfrac{1}{2} \sqrt{2\omega_{\bar{k}}}\, e^{-i\omega_{\bar{k}} t} \int d^3\bar{x}\, e^{+i\bar{k}.\bar{x}}\, \varepsilon_{\mu}(\bar{k},\lambda) \left\{ A^{\mu} - \frac{i}{\omega_{\bar{k}}} \dot{A}^{\mu} \right\}$$

$\hookrightarrow \left[a(\bar{k},\lambda),\, a^{+}(\bar{k}',\lambda') \right]_{-}$

$$= \zeta_{\lambda} \zeta_{\lambda'} \frac{1}{4} (2\omega_{\bar{k}})^{1/2} (2\omega_{\bar{k}'})^{1/2} e^{i(\omega_{\bar{k}} - \omega_{\bar{k}'})t}\, \varepsilon^{*}_{\mu}(\bar{k},\lambda)\, \varepsilon_{\nu}(\bar{k}',\lambda')$$

$$\cdot \int d^3\bar{x} \int d^3\bar{x}'\, e^{-i\bar{k}.\bar{x}}\, e^{+i\bar{k}'.\bar{x}'}$$

$$\cdot \underbrace{\left[A^{\mu}(\bar{x},t) + \frac{i}{\omega_{\bar{k}}} \dot{A}^{\mu}(\bar{x},t),\, A^{\nu}(\bar{x}',t) - \frac{i}{\omega_{\bar{k}'}} \dot{A}^{\nu}(\bar{x}',t) \right]_{-}}_{-\frac{1}{\omega_{\bar{k}'}} g^{\mu\nu} \delta^3(\bar{x}-\bar{x}') - \frac{1}{\omega_{\bar{k}}} g^{\mu\nu} \delta^3(\bar{x}-\bar{x}')}$$

$$= \zeta_{\lambda} \zeta_{\lambda'} \frac{1}{4} (2\omega_{\bar{k}})^{1/2} (2\omega_{\bar{k}'})^{1/2} (-1) \left(\frac{1}{\omega_{\bar{k}}} + \frac{1}{\omega_{\bar{k}'}} \right) e^{i(\omega_{\bar{k}} - \omega_{\bar{k}'})t}$$

$$\cdot\, \varepsilon^{*}_{\mu}(\bar{k},\lambda)\, \varepsilon^{\mu}(\bar{k}',\lambda') \underbrace{\int d^3\bar{x}\, e^{-i(\bar{k}-\bar{k}').\bar{x}}}_{(2\pi)^3 \delta^3(\bar{k}-\bar{k}')}$$

$$\left[a(\bar{k},\lambda) , a^{+}(\bar{k}',\lambda') \right]_{-}$$

$$= - \zeta_\lambda \zeta_{\lambda'} \frac{1}{4} (2\omega_{\bar{k}}) \left(\frac{2}{\omega_{\bar{k}}} \right) (2\pi)^3 \underbrace{\varepsilon_\mu^*(\bar{k},\lambda) \varepsilon^\mu(\bar{k},\lambda')}_{-\zeta_\lambda \delta_{\lambda\lambda'}}$$

$$\downarrow \quad \zeta_\lambda^2 = 1$$

$$= \zeta_\lambda \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\bar{k}-\bar{k}')$$

$$\therefore \quad \boxed{\begin{array}{l} \left[a(\bar{k},\lambda), a^{+}(\bar{k}',\lambda') \right]_{-} = \zeta_\lambda \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\bar{k}-\bar{k}') \\ \text{ANALOGOUSLY} \\ \left[a(\bar{k},\lambda), a(\bar{k}',\lambda') \right]_{-} = 0 \\ \left[a^{+}(\bar{k},\lambda), a^{+}(\bar{k}',\lambda') \right]_{-} = 0 \end{array}}$$

$$\zeta_0 = -1, \quad \zeta_i = +1 \quad (i = 1,2,3)$$

FOR $\lambda = 1, 2, 3$

$$\left[a(\bar{k},\lambda), a^{+}(\bar{k}',\lambda) \right]_{-} = \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\bar{k}-\bar{k}')$$

FOR $\lambda = 0$

$$\left[a(\bar{k},0), a^{+}(\bar{k}',0) \right]_{-} = - (2\pi)^3 \delta^3(\bar{k}-\bar{k}')$$

FOR $\lambda = 1,2,3 \longrightarrow$ STANDARD BOSON COMMUTATION RELATIONS

$\lambda = 0 \longrightarrow$ SIGN CHANGE

↳ IF WE CONSIDER VACUUM STATE AS STATE WITHOUT PHOTONS

i.e. $a(\bar{k},\lambda)\,|0\rangle = 0 \qquad \lambda = 0,1,2,3$

$\Downarrow$

1 - PHOTON STATE WITH POLARIZATION λ

$$|1\gamma,\lambda\rangle = \int d^3\bar{k}\; f(\bar{k})\; a^+(\bar{k},\lambda)\,|0\rangle$$

↳ WAVE PACKET

$$\int d^3\bar{k}\; |f(\bar{k})|^2 < \infty$$

NORMALIZATION OF 1γ STATE

$$\langle 1\gamma,\lambda\,|\,1\gamma,\lambda\rangle$$

$$= \int d^3\bar{k} \int d^3\bar{k}'\; f^*(\bar{k})\, f(\bar{k}')\;\cdot \underbrace{\langle 0|\, a(\bar{k},\lambda)\, a^+(\bar{k}',\lambda)\,|0\rangle}_{\zeta_\lambda\,(2\pi)^3\,\delta^3(\bar{k}-\bar{k}')}$$

$$= \zeta_\lambda\,(2\pi)^3 \int d^3\bar{k}\; |f(\bar{k})|^2$$

$$= \begin{cases} > 0 & \text{FOR } \lambda = 1,2,3 \\ < 0 & \text{FOR } \lambda = 0 \;! \quad \text{NEGATIVE NORM STATE} \end{cases}$$

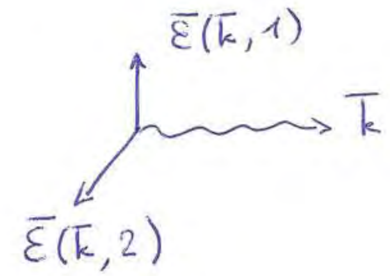
↳ HAMILTONIAN :

$$H = \int \frac{d^3\bar{k}}{(2\pi)^3} \, \omega_{\bar{k}} \left\{ - a^+(\bar{k},0)\, a(\bar{k},0) + \sum_{i=1}^{3} a^+(\bar{k},i)\, a(\bar{k},i) \right\}$$

STATES WITH $\lambda = 0$ LEAD TO NEGATIVE ENERGY !

↳ PHYSICAL STATES :

FREE MAXWELL FIELD HAS ONLY 2 TRANSVERSE COMPONENTS



↳ $\lambda = 0$, $\lambda = 3$ STATES SHOULD NOT APPEAR IN PHYSICAL STATES UPON QUANTIZATION

$|\Psi\rangle$ PHYSICAL VACUUM (HAS NO TRANSVERSE PHOTONS)

WE REQUIRE

$$\underline{\underline{\partial_\mu A^{\mu(+)} |\Psi\rangle = 0}} \qquad \text{(GUPTA-BLEULER)}$$

(+) STANDS FOR POS. FREQUENCY (ANNIHILATING) PART IN A^μ

NOTE : THIS IS A WEAKER CONDITION THAN CONSIDERING $\partial_\mu \hat{A}^\mu = 0$ AS OPERATOR CONDITION

$$\partial_\mu A^{\mu(+)} \, | \Psi \rangle = 0$$

$$\Downarrow$$

$$\partial_\mu \int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda a(\bar{k},\lambda)\, \varepsilon^\mu(\bar{k},\lambda)\, e^{-ik.x} \, | \Psi \rangle = 0$$

$$\int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \sum_\lambda (-i)\, a(\bar{k},\lambda)\, e^{-ik.x} \, k_\mu \varepsilon^\mu(\bar{k},\lambda) \, | \Psi \rangle = 0$$

$$\Downarrow \qquad \begin{aligned} & k_\mu \varepsilon^\mu(\bar{k},\lambda) \\ &= \omega_{\bar{k}}\, \varepsilon^0(\bar{k},\lambda) - |\bar{k}|\, \varepsilon^3(\bar{k},\lambda) \\ &= \omega_{\bar{k}} \left(\delta_{\lambda 0} - \delta_{\lambda 3} \right) \end{aligned}$$

$$\forall \bar{k} : \quad \underline{\underline{\left(a(\bar{k},0) - a(\bar{k},3) \right) | \Psi \rangle = 0 .}}$$

$$\Downarrow$$

$$\langle \Psi | a^\dagger(\bar{k},3)\, a(\bar{k},3) | \Psi \rangle = \langle \Psi | a^\dagger(\bar{k},0)\, a(\bar{k},0) | \Psi \rangle$$

- PHYSICAL VACUUM STATE HAS EQUAL NUMBER OF SCALAR ($\lambda = 0$) AND LONGITUDINAL ($\lambda = 3$) PHOTONS

- THE COMBINED ENERGY OF SCALAR & LONGITUDINAL PHOTONS IS ZERO

$$\langle \Psi \,|\, H \,|\, \Psi \rangle = \int \frac{d^3\bar{k}}{(2\pi)^3}\, \omega_{\bar{k}} \sum_{\lambda=1,2} \langle \Psi \,|\, a^{+}(\bar{k},\lambda)\, a(\bar{k},\lambda) \,|\, \Psi \rangle$$

- PHYSICAL QUANTITIES ONLY INVOLVE TRANSVERSE PHOTONS ($\lambda = 1, 2$).

- ALTERING THE ALLOWED ADMIXTURES OF SCALAR AND LONGITUDINAL PHOTONS IS EQUIVALENT TO A GAUGE TF. BETWEEN 2 POTENTIALS, BOTH OF WHICH ARE IN LORENZ GAUGE

  ($\rightarrow$ EXERCISE)

3) FEYNMAN PROPAGATOR FOR PHOTONS

$\Rightarrow$ 2 POINT CORRELATION FUNCTIONS.

$\hookrightarrow$ $\langle 0 | A^\mu(x) A^\nu(y) | 0 \rangle = i D^{\mu\nu}(x-y)$

WITH $|0\rangle$ STATE WITHOUT ANY PHOTONS.

$$i D^{\mu\nu}(x-y)$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3 \sqrt{2\omega_{\bar{k}}}} \int \frac{d^3\bar{k}'}{(2\pi)^3 \sqrt{2\omega_{\bar{k}'}}} \sum_{\lambda=0}^{3} \sum_{\lambda'=0}^{3}$$

$$\langle 0 | \left(a(\bar{k},\lambda) \varepsilon^\mu(\bar{k},\lambda) e^{-ik.x} + \cancel{a^+(\bar{k},\lambda)} \varepsilon^{*\mu}(\bar{k},\lambda) e^{+ik.x} \right)$$

$$\cdot \left(\cancel{a(\bar{k}',\lambda')} \varepsilon^\nu(\bar{k}',\lambda') e^{-ik'.y} + a^+(\bar{k}',\lambda') \varepsilon^{*\nu}(\bar{k}',\lambda') e^{+ik'.y} \right) |0\rangle$$

$$= \int \frac{d^3\bar{k}}{(2\pi)^3} \frac{1}{2\omega_{\bar{k}}} \sum_{\lambda=0}^{3} \zeta_\lambda \, \varepsilon^\mu(\bar{k},\lambda) \varepsilon^{*\nu}(\bar{k},\lambda) \, e^{-ik.(x-y)} \Big|_{k^0=\omega_{\bar{k}}}$$

↳ POLARIZATION SUM

$$\boxed{\sum_{\lambda=0}^{3} \zeta_\lambda \, \varepsilon^{\mu}(\bar{k},\lambda)\, \varepsilon^{*\nu}(\bar{k},\lambda) = -g^{\mu\nu}}$$

FOR CHOICE $k^{\mu}(|\bar{k}|, 0, 0, |\bar{k}|)$

$$\varepsilon^{\mu}(\bar{k},\lambda) = g^{\mu}_{\lambda}$$

$$\sum_{\lambda=0}^{3} \zeta_\lambda \, \varepsilon^{\mu}(\bar{k},\lambda)\, \varepsilon^{*\nu}(\bar{k},\lambda)$$

$$= - g^{\mu}_{0} g^{\nu}_{0} + g^{\mu}_{1} g^{\nu}_{1} + g^{\mu}_{2} g^{\nu}_{2} + g^{\mu}_{3} g^{\nu}_{3}$$

FOR $\mu \neq \nu \rightarrow 0$

$\mu = \nu = 0 \rightarrow -1 = -g^{00}$

$\mu = \nu = i \rightarrow +1 = -g^{ii}$
$(1,2,3)$

↳ $i D^{\mu\nu}(x-y)$

$$= (-g^{\mu\nu}) \int \frac{d^3\bar{k}}{(2\pi)^3 2\omega_{\bar{k}}} \, e^{-ik.(x-y)} \Big|_{k^0 = \omega_{\bar{k}}}$$

$\underline{\underline{\omega_{\bar{k}} = |\bar{k}|}}$

$$\boxed{D^{\mu\nu}(x) = (-g^{\mu\nu})\, \Delta(x)}$$

↳ 2 POINT CORRELATOR FOR KLEIN-GORDON THEORY (WITH $m = 0$)

⇒ FEYNMAN PROPAGATOR

↳ $$\boxed{\langle 0 | T\, A^\mu(x)\, A^\nu(y) | 0 \rangle \equiv i D_F^{\mu\nu}(x-y)}$$

WITH (BOSON FIELDS)

$$T\, A^\mu(x) A^\nu(y) = \theta(x^0 - y^0)\, A^\mu(x)\, A^\nu(y) + \theta(y^0 - x^0)\, A^\nu(y)\, A^\mu(x).$$

↳ $$\boxed{D_F^{\mu\nu}(x) = (-g^{\mu\nu})\, \Delta_F(x)}$$

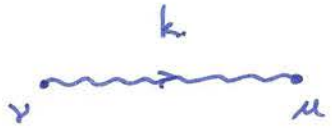
WITH $\Delta_F(x) = \theta(x^0)\, \Delta(x) + \theta(-x^0)\, \Delta(-x)$

↳ FEYNMAN PROPAGATOR FOR SPIN-0 PARTICLE WITH $m = 0$

↳ IN MOMENTUM SPACE

$$D_F^{\mu\nu}(x) = \int \frac{d^4 k}{(2\pi)^4}\, e^{-ik.x}\, \tilde{D}_F^{\mu\nu}(k)$$

$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\varepsilon}}$$



FEYNMAN PROPAGATOR FOR PHOTON IN LORENZ GAUGE

(ALSO CALLED FEYNMAN GAUGE)

⇒ MORE GENERAL GAUGES

↳ $\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{G.F.}$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2$$

<u>NOTE</u> : CHOICE $\xi = 1$ CORRESPONDS WITH LORENZ (FEYNMAN) GAUGE

↳ FIELD EQUATIONS.

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi} \partial^\nu \left(\partial_\mu A^\mu\right) = 0$$

$$\Box A^\nu - \left(1 - \frac{1}{\xi}\right) \partial^\nu \left(\partial_\mu A^\mu\right) = 0.$$

↳ PROPAGATOR

$$\underbrace{\left[-k^2 g^{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k^\mu k^\nu\right]}_{} A_\mu = 0$$

PROPAGATOR IS INVERSE OF THIS OPERATOR

$$\left(-k^2 g^{\mu\nu} + \left(1-\frac{1}{\xi}\right) k^\mu k^\nu\right)\left(A\, g_{\nu\lambda} + B\, k_\nu k_\lambda\right) = g^\mu_\lambda$$

$$g^\mu_\lambda = -A k^2 g^\mu_\lambda + \left[(A + Bk^2)\left(1-\frac{1}{\xi}\right) - Bk^2\right] k^\mu k_\lambda$$

$$\Downarrow$$

$$\begin{cases} -Ak^2 = 1 \\ (A + Bk^2)\left(1-\frac{1}{\xi}\right) - Bk^2 = 0 \end{cases}$$

$$A = -1/k^2$$

$$A\left(1-\frac{1}{\xi}\right) = \frac{Bk^2}{\xi} \Rightarrow B = -\frac{(1-\xi)}{k^2} A$$

PHOTON FEYNMAN PROPAGATOR

$$\boxed{\tilde{D}_F^{\mu\nu}(k) = \frac{1}{k^2 + i\varepsilon}\left(-g^{\mu\nu} + (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)}$$