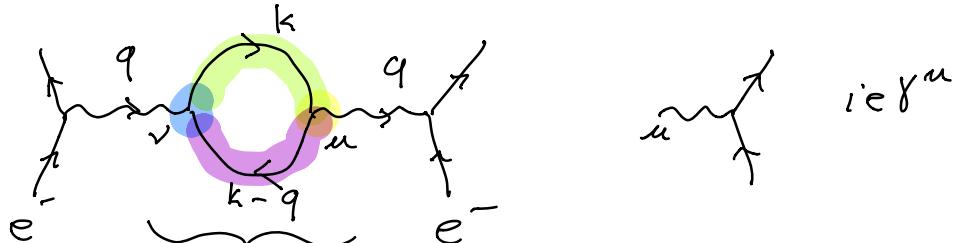


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LECTURE 20

VACUUM POLARIZATION in SPINOR QED



$$\sigma^{\mu\nu} = i \bar{\Pi}^{\mu\nu}(q)$$

$$i \bar{\Pi}^{\mu\nu}(q) = (-i) \int \frac{d^4 k}{(2\pi)^4} \frac{\text{Tr} \{ (ie\gamma^\mu) i(k+m) (ie\gamma^\nu) i(k-q+m) \}}{[k^2 - m^2 + i\varepsilon] [(k-q)^2 - m^2 + i\varepsilon]}$$

$$\bar{\Pi}^{\mu\nu}(q) = ie^2 \int \frac{d^D k}{(2\pi)^D} \underbrace{\frac{\text{Tr} \{ \gamma^\mu (k+m) \gamma^\nu (k-q+m) \}}{[...][...]}_A}_B$$

$$\frac{1}{[A+(B-A)x]^2} \rightarrow B-A = -2k \cdot q + q^2$$

$$k \rightarrow k' = k - qx$$

$$k = k' + qx$$

$$(k-qx)^2 + q^2 x (1-x) - m^2$$

$$= ie^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{\text{Tr} \{ \gamma^\mu (ks + qx + m) \gamma^\nu (k - q(1-x) + m) \}}{[k^2 - \Delta]^2}$$

$$\Delta = m^2 - q^2 x (1-x) - i\varepsilon$$

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$$\downarrow \quad \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^2} = 0$$

$$= ie^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^2}$$

$$\cdot \left\{ \text{Tr} \left\{ \underbrace{\gamma^\mu \gamma^\nu}_{\gamma^\mu \gamma^\nu} \right\} \right\}$$

$$- \times (1-x) \text{Tr} \left\{ \gamma^\mu \gamma^\nu \right\} + m^2 \text{Tr} \left\{ \gamma^\mu \gamma^\nu \right\}$$

DIRAC ALGEBRA IN D DIM

$$\rightsquigarrow \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \quad \text{(II)}$$

$$\underbrace{\gamma_\mu g_{\mu\nu}}_{\propto g_{\mu\nu}} \quad \cancel{\gamma_\mu \gamma^\mu} = \cancel{2 \underbrace{g_{\mu\nu} g^{\mu\nu}}_D} \quad \text{II}$$

$$\epsilon = 2 - D/2 \quad \epsilon = 0$$

$$\begin{aligned} \underbrace{\gamma_\mu \gamma^\alpha \gamma^\mu}_{\gamma_\mu \gamma^\alpha} &= (2g_\mu^\alpha - \gamma^\alpha \gamma_\mu) \gamma^\mu \\ &= 2\gamma^\alpha - \gamma^\alpha D \\ &= (2-D) \gamma^\alpha \end{aligned}$$

γ^μ : MATRICES IN $f(D)$ DIMENSION

$$f(D=4) = 4$$

$$f(D) = 4 + \underbrace{(4-D)}_{-2\epsilon} f'(4) + O(\epsilon^2)$$

PHYS. RESULT WILL NOT DEPEND
ON $\oint \gamma^i d\epsilon = 0$

$$\text{Tr} \{ \mathbb{I} \} = g(D)$$

$$\text{Tr} \{ \gamma^\mu \gamma^\nu \} = g(D) g^{\mu\nu}$$

$$\text{Tr} \{ \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu \} = g(D) \left\{ g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu} \right\}$$

$$\Pi^{\mu\nu}(q) = i e^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^2}$$

$$g(D) \left\{ \begin{aligned} & \frac{2}{-x(1-x)} \left(\frac{k^\mu k^\nu}{m^2} - \frac{k^2}{m^2} g^{\mu\nu} \right. \\ & \left. + \frac{q^2}{m^2} g^{\mu\nu} \right) \\ & - 2 \times (1-x) q^\mu q^\nu \end{aligned} \right.$$

$$\rightarrow \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^2} = \frac{i}{(4\pi)^2} (4\pi)^\varepsilon \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2 - \Delta)^2} = \frac{-i g^{\mu\nu}}{(4\pi)^2} (4\pi)^\varepsilon \frac{\Gamma(-1+\varepsilon)}{2\Delta^{-1+\varepsilon}}$$

$$\Pi^{\mu\nu}(q) = ie^2 \int_0^1 dx \frac{i}{(4\pi)^2} (4\pi\mu^2)^\varepsilon \cdot g(D)$$

$$\cdot \left\{ -g^{\mu\nu} \left(\frac{(2-D)}{2} \right) \frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}} \right.$$

$$1 - \frac{D}{2} = -1 + \varepsilon$$

$$\left. (-1+\varepsilon) \Gamma(-1+\varepsilon) = \Gamma(\varepsilon) \right\}$$

$$+ \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon} \left[g^{\mu\nu} (m^2 + q^2 \times (1-x)) - 2 \times (1-x) q^\mu q^\nu \right] \}$$

$$= -\frac{e^2}{(4\pi)^2} \Gamma(\varepsilon) (4\pi\mu^2)^\varepsilon \int_0^1 dx \frac{1}{\Delta^\varepsilon}$$

$$\cdot \left\{ -g^{\mu\nu} (\cancel{m^2} - q^2 \times (1-x)) \right.$$

$$+ g^{\mu\nu} (\cancel{m^2} + q^2 \times (1-x))$$

$$\left. - 2 \times (1-x) q^\mu q^\nu \right\}$$

$$\Pi(q) = -\frac{e^2}{(4\pi)^2} \Gamma(\varepsilon) (4\pi\mu^2)^\varepsilon g(D) \int_0^1 dx \frac{1}{\Delta^\varepsilon}$$

$$2 \times (1-x) (q^2 g^{\mu\nu} - q^\mu q^\nu)$$

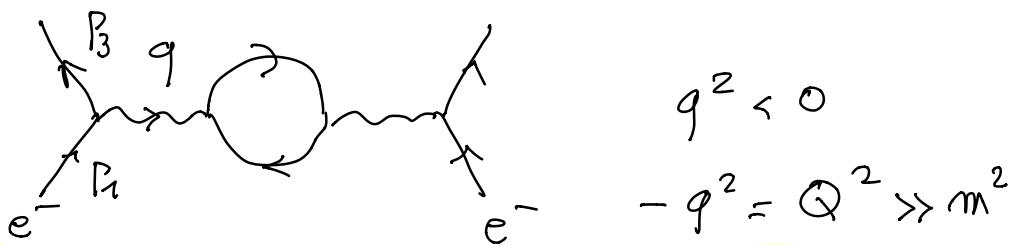
$$q_\mu \Pi^{\mu\nu} = 0 \quad , \quad q_\nu \Pi^{\mu\nu} = 0$$

GAUGE INV.

$$\Pi^{uv}(q) = - (q^2 g^{uv} - q^u q^v) \frac{e^2}{8\pi^2} \cdot \underbrace{\left(\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) \right)}_{\text{1}} \cdot \int_0^1 dx \times (1-x) \left[1 + \epsilon \ln \frac{u^2}{m^2 - q^2 x(1-x)} \right] \uparrow$$

$$\int dx (1-x) x = \frac{1}{6}$$

$$\Pi^{uv}(q) = - (q^2 g^{uv} - q^u q^v) \frac{e^2}{48\pi^2} \cdot 4 \left(\frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \frac{1}{2} \text{f}'(\epsilon) \right) + 6 \int_0^1 dx \times (1-x) \ln \frac{u^2}{m^2 - q^2 x(1-x)}$$



$\boxed{\Pi^{uv}(q) \xrightarrow{Q^2 \gg m^2} - (q^2 g^{uv} - q^u q^v) \frac{e^2}{12\pi^2} \left\{ \frac{1}{\epsilon} - \gamma_E + \ln(4\pi) - \frac{1}{2} \text{f}'(\epsilon) + \ln \frac{u^2}{Q^2} - 6 \int_0^1 dx \times (1-x) \cdot \ln x(1-x) \right\}}$

SPINOR QED

513

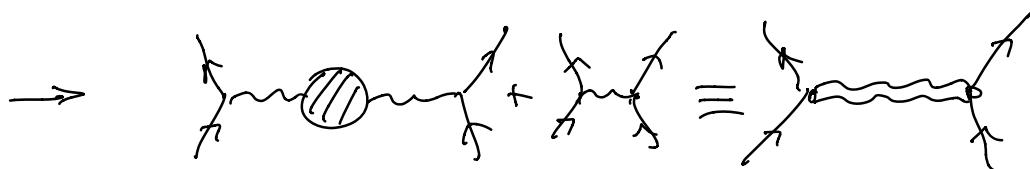
\Rightarrow SCALAR QED

$$\Pi^{\mu\nu}(q) \xrightarrow{Q^2 \gg m^2} -\frac{e^2}{48\pi^2} \left(q^\mu q^\nu - q^\nu q^\mu \right)$$

$$+ \left\{ \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln \frac{el^2}{Q^2} \right.$$

$$\left. + \frac{8}{3} \right\}$$

\hookrightarrow 'DRESSED' PHOTON PROPAGATOR



$$\text{---} = \text{---} + \text{---}$$

$$\mathcal{O}(e^2) \quad \mathcal{O}(e^4)$$

+

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$$\begin{aligned} \tilde{\omega} &= \omega + \text{[Diagram: wavy line with a yellow shaded loop attached]} \\ &\quad + \text{[Diagram: wavy line with two yellow shaded loops attached]} \\ &\quad + \dots \\ \tilde{\omega}^q &= \omega + \text{[Diagram: wavy line with a yellow shaded loop attached, labeled } \beta \text{ and } \alpha \text{ below it]} \end{aligned}$$

DYSON EXPANSION

$$iD^{uv}(q) = \underbrace{iD_0^{uv}(q)}_{\sim} + iD_0^{u\alpha}(\overset{\beta}{q}) i\Pi_{\alpha\beta}^{v\gamma}(\overset{\beta}{q}) \cdot iD_\gamma(q)$$

$$- \frac{g^{uv}}{q^2 + i\varepsilon}$$

$$\begin{aligned} \Pi^{uv}(q) &= (q^2 g^{uv} - q^u q^v) \Pi(q^2) \\ &= \underbrace{(g^{uv} - \frac{q^u q^v}{q^2})}_{\equiv \Delta^{uv}(q)} q^2 \Pi(q^2) \end{aligned}$$

$$\begin{aligned} \overline{\Pi}_{\alpha\beta} D_\alpha^{\beta\gamma} &= \Delta_{\alpha\beta} \cancel{q^2 \Pi(q^2)} \cancel{- \frac{q^{\beta\gamma}}{q^2}} \\ &= - \Delta_{\alpha}^{\gamma} \Pi(q^2) \end{aligned}$$

$$\begin{aligned}
 D^{\mu\nu} &= D_0^{\mu\nu} - D_0^{\mu\alpha} \overline{\Pi}_{\alpha\beta} D_0^{\beta\nu} \\
 &\quad + D_0^{\mu\alpha} \overline{\Pi}_{\alpha\gamma} D_0^{\gamma\nu} \overline{\Pi}_{\gamma\delta} D_0^{\delta\nu} \\
 &\quad + \dots \\
 &= D_0^{\mu\nu} + D_0^{\mu\alpha} \Delta_\alpha^\nu \overline{\Pi} \\
 &\quad + D_0^{\mu\alpha} \Delta_\alpha^\gamma \Delta_\gamma^\nu \overline{\Pi}^2 \\
 &\quad + \dots
 \end{aligned}$$

$$\Delta_\alpha^\gamma = \left(g_\alpha^\gamma - \frac{g_\alpha g^\gamma}{g^2} \right)$$

$$\begin{aligned}
 \Delta_\alpha^\gamma \Delta_\gamma^\nu &\stackrel{?}{=} \left(g_\alpha^\gamma - \frac{g_\alpha g^\gamma}{g^2} \right) \left(g_\gamma^\nu - \frac{g_\gamma g^\nu}{g^2} \right) \\
 &= g_\alpha^\nu - \frac{g_\alpha g^\nu}{g^2} \\
 &= \Delta_\alpha^\nu
 \end{aligned}$$

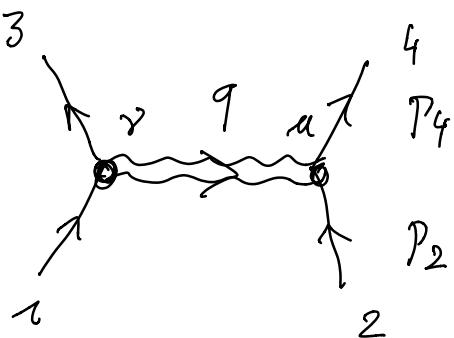
$$D^{\mu\nu} = D_0^{\mu\nu} + D_0^{\mu\alpha} \Delta_\alpha^\nu \left[\overline{\Pi} + \overline{\Pi}^2 + \dots \right]$$

$$\begin{aligned}
 &= -\frac{g^{uv}}{q^2} - \frac{\left(g^{uv} - \frac{q^u q^v}{q^2}\right)}{q^2} \left[\pi + \overline{\pi}^2 \right. \\
 &\quad \left. + \dots \right] \\
 &= -\frac{1}{q^2} \left(g^{uv} - \frac{q^u q^v}{q^2} \right) \left[1 + \overline{\pi} + \overline{\pi}^2 \right. \\
 &\quad \left. + \dots \right] \\
 &\quad - \frac{q^u q^v}{q^4}
 \end{aligned}$$

$$\frac{1}{1 - \overline{\pi}(q^2)}$$

$$\boxed{
 \begin{aligned}
 D^{uv}(q) &= \frac{1}{q^2 [1 - \overline{\pi}(q^2)]} \left(-g^{uv} + \frac{q^u q^v}{q^2} \right) \\
 &\quad - \frac{q^u q^v}{q^4}
 \end{aligned}
 }$$

$$\frac{e^2}{4\pi} = \kappa_{em} = \frac{1}{137}$$



$$q = P_4 - P_2$$

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$$\mathcal{M} = \bar{U}_3 (ie\gamma_5) U_1 \quad \bar{U}_4 (ie\gamma_5) U_2$$

$$\cdot \quad i D^{\mu\nu}(q)$$

$$\left. \begin{array}{l} q_\nu \bar{U}_3 \gamma^\nu U_1 = 0 \end{array} \right.$$

$$\left. \begin{array}{l} q_\alpha \bar{U}_4 \gamma^\alpha U_2 = 0 \end{array} \right.$$

$$\hookrightarrow \bar{U}_4 (\cancel{P}_4 - \cancel{P}_2) U_2 = \bar{U}_4 (m - m) \quad \begin{matrix} \curvearrowleft & \curvearrowright \\ m & m \end{matrix} \quad \stackrel{!}{=} 0 \quad \begin{matrix} & \\ & U_1 \end{matrix}$$

$$\mathcal{M} = + ie^2 \bar{U}_3 \gamma^\mu U_1 \quad \bar{U}_4 \gamma_\mu U_2$$

$$\cdot \quad \frac{1}{q^2 [1 - \Pi(q^2)]}$$

$$V_C(q^2) = \frac{e^2}{q^2}$$

INCLUDING VACUUM POL.

$$V_C(q^2) \rightarrow V(q^2) = \frac{e^2}{q^2 [1 - \Pi(q^2)]}$$

$$\Pi(q^2) = \Pi(0) + q^2 \Pi'(0) + \dots$$

$$1 - \Pi(q^2) = [1 - \Pi(0)] \left\{ 1 - q^2 \frac{\Pi'(0)}{1 - \Pi(0)} + O(q^4) \right\}$$

$$V(q^2) = \frac{e^2}{q^2 [1 - \Pi(0)]} \left\{ 1 + q^2 \frac{\Pi'(0)}{1 - \Pi(0)} + O(q^4) \right\}$$

III
 $\frac{e_R^2}{4\pi} = \frac{1}{137}$