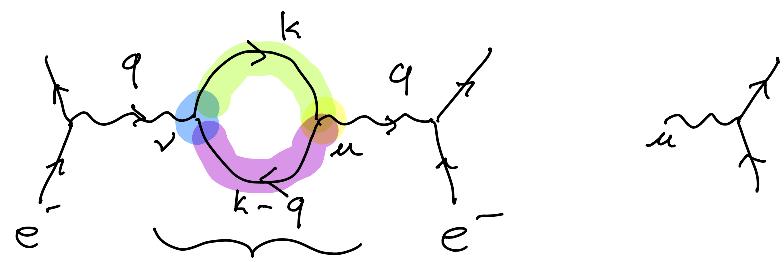
⟹ LECTURE 20

VACUUM POLARIZATION IN SPINOR QED



$ie\gamma^\mu$

$$\mathcal{M}^{\mu\nu} \equiv i\Pi^{\mu\nu}(q)$$

$$i\Pi^{\mu\nu}(q) = (-1)\int \frac{d^4k}{(2\pi)^4}\, \frac{Tr\{(ie\gamma^\mu)\, i(\not k+m)\,(ie\gamma^\nu)\, i(\not k-\not q+m)\}}{[k^2-m^2+i\varepsilon][(k-q)^2-m^2+i\varepsilon]}$$

$$\Pi^{\mu\nu}(q) = ie^2 \int \frac{d^{\cancel{4}\,D}k}{(2\pi)^{\cancel{4}\,D}}\, \frac{Tr\{\gamma^\mu(\not k+m)\gamma^\nu(\not k-\not q+m)\}}{\underbrace{[\quad\dots\quad]}_{A}\ \underbrace{[\quad\dots\quad]}_{B}}$$

$$\frac{1}{[A+(B-A)x]^2}$$

$$B-A = -2k\cdot q+q^2$$

$$k^2-m^2+(-2k\cdot q+q^2)x$$

$$(k-qx)^2+q^2x(1-x)-m^2$$

$$k \to k' = k-qx$$

$$k = k'+qx$$

$$= ie^2\int_0^1 dx \int \frac{d^Dk}{(2\pi)^D}\, \frac{Tr\{\gamma^\mu(\not k+\not q x+m)\gamma^\nu(\not k-\not q(1-x)+m)\}}{[k^2-\Delta]^2}$$

$$\Delta = m^2-q^2x(1-x)-i\varepsilon$$

$$\downarrow \qquad \int \frac{d^Dk}{(2\pi)^D} \frac{\not{k}}{(k^2-\Delta)^2} = 0$$

$$= ie^2 \int_0^1 dx \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-\Delta)^2}$$

$$\cdot \Big\{ \mathrm{Tr}\{\gamma^\mu \not{k}\, \gamma^\nu \not{k}\}$$

$$- x(1-x)\, \mathrm{Tr}\{\gamma^\mu \not{q}\, \gamma^\nu \not{q}\} + m^2\, \mathrm{Tr}\{\gamma^\mu \gamma^\nu\} \Big\}$$

DIRAC ALGEBRA IN D DIM

$$\rightsquigarrow \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}\, \mathbb{1}$$

$$\hookrightarrow \times g_{\mu\nu} \qquad \cancel{2}\, \gamma_\mu \gamma^\mu = \cancel{2}\, \underbrace{g_{\mu\nu}\, g^{\mu\nu}}_{D}\, \mathbb{1}$$

$$\varepsilon = 2 - D/2 \qquad \varepsilon = 0$$

$$\gamma_\mu \gamma^\alpha \gamma^\mu = \left(2 g_\mu^{\ \alpha} - \gamma^\alpha \gamma_\mu\right) \gamma^\mu$$

$$= 2\gamma^\alpha - \gamma^\alpha D$$

$$= (2 - D)\, \gamma^\alpha$$

γ^μ : MATRICES IN $f(D)$ DIMENSION

$$f(D=4) = 4$$

$$f(D) = 4 + \underbrace{(4-D)}_{-2\varepsilon} f'(4) + O(\varepsilon^2)$$

PHYS. RESULT WILL NOT DEPEND ON $f'(4)$

$$= 0$$

$$\mathrm{Tr}\{\mathbb{1}\} = f(D)$$

$$\mathrm{Tr}\{\gamma^\mu \gamma^\nu\} = f(D)\, g^{\mu\nu}$$

$$\mathrm{Tr}\{\gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu\} = f(D)\{g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} + g^{\alpha\nu} g^{\beta\mu}\}$$

$$\Pi^{\mu\nu}(q) = i e^2 \int_0^1 dx \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta)^2}$$

$$f(D)\{2 k^\mu k^\nu - k^2 g^{\mu\nu} - x(1-x)(2 q^\mu q^\nu - q^2 g^{\mu\nu}) + m^2 g^{\mu\nu}\}$$

$$g^{\mu\nu}(m^2 + q^2 x(1-x)) - 2x(1-x)\, q^\mu q^\nu$$

$$\longrightarrow \int \frac{d^D k}{(4\pi)^D} \frac{1}{(k^2-\Delta)^2} = \frac{i}{(4\pi)^2} (4\pi)^\varepsilon \frac{\Gamma(\varepsilon)}{\Delta^\varepsilon}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2-\Delta)^2} = \frac{-i g^{\mu\nu}}{(4\pi)^2} (4\pi)^\varepsilon \frac{\Gamma(-1+\varepsilon)}{2\Delta^{-1+\varepsilon}}$$

$$\Pi^{\mu\nu}(q) = ie^2 \int_0^1 dx \ \frac{i}{(4\pi)^2} \ (4\pi\mu^2)^{\varepsilon} \cdot g(D)$$

$$\cdot \left\{ - g^{\mu\nu} \left(\frac{2-D}{2}\right) \frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}} \right.$$

$$1 - \frac{D}{2} = -1 + \varepsilon$$

$$(-1+\varepsilon)\,\Gamma(-1+\varepsilon) = \Gamma(\varepsilon)$$

$$\left. + \frac{\Gamma(\varepsilon)}{\Delta^{\varepsilon}} \left[g^{\mu\nu}\left(m^2 + q^2 x(1-x)\right) - 2x(1-x)\, q^\mu q^\nu \right] \right\}$$

$$= -\frac{e^2}{(4\pi)^2} \Gamma(\varepsilon) (4\pi\mu^2)^{\varepsilon} \int_0^1 dx \ \frac{1}{\Delta^{\varepsilon}}$$

$$\cdot \left\{ - g^{\mu\nu}\left(\cancel{m^2} - q^2 x(1-x)\right) + g^{\mu\nu}\left(\cancel{m^2} + q^2 x(1-x)\right) - 2x(1-x)\, q^\mu q^\nu \right\}$$

$$\Pi^{\mu\nu}(q) = -\frac{e^2}{(4\pi)^2} \Gamma(\varepsilon) (4\pi\mu^2)^{\varepsilon} g(D) \int_0^1 dx \ \frac{1}{\Delta^{\varepsilon}}$$

$$2x(1-x)\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)$$

$$q_\mu \Pi^{\mu\nu} = 0 \quad , \quad q_\nu \Pi^{\mu\nu} = 0$$

GAUGE INV.

$$\Pi^{\mu\nu}(q) = -\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \frac{e^2}{8\pi^2} \cdot \underbrace{f(D)}_{4 - 2\varepsilon f'(4)}$$

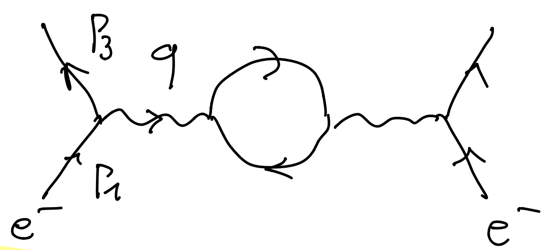
$$\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi)\right)$$

$$\cdot \int_0^1 dx \; x(1-x) \left[1 + \varepsilon \ln \frac{\mu^2}{m^2 - q^2 x(1-x)}\right]$$

$$\int dx\,(1-x)\,x = \frac{1}{6}$$

$$\Pi^{\mu\nu}(q) = -\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \frac{e^2}{48\pi^2} \cdot 4$$

$$\left(\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \frac{1}{2} f'(4) + 6 \int_0^1 dx \; x(1-x) \ln \frac{\mu^2}{m^2 - q^2 x(1-x)}\right)$$



$$q^2 < 0$$

$$-q^2 = Q^2 \gg m^2$$

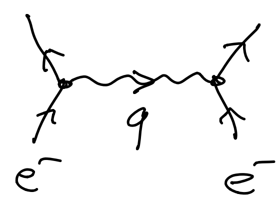
$$\boxed{\Pi^{\mu\nu}(q) \xrightarrow[Q^2 \gg m^2]{} -\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \frac{e^2}{12\pi^2} \left\{\frac{1}{\varepsilon} - \gamma_E + \ln(4\pi) - \frac{1}{2} f'(4) + \ln \frac{\mu^2}{Q^2} - \underbrace{6 \int_0^1 dx \; x(1-x) \cdot \ln x(1-x)}_{5/3}\right\}}$$

SPINOR QED

$\Rightarrow$ SCALAR QED

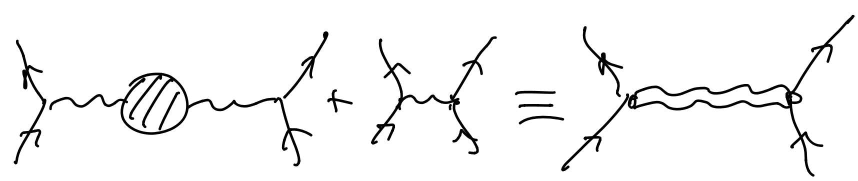
$$\boxed{\Pi^{\mu\nu}(q) \xrightarrow[\underset{\sim\sim\sim}{Q^2 \gg m^2}]{} -\frac{e^2}{48\pi^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{Q^2} + \frac{8}{3} \right\}}$$

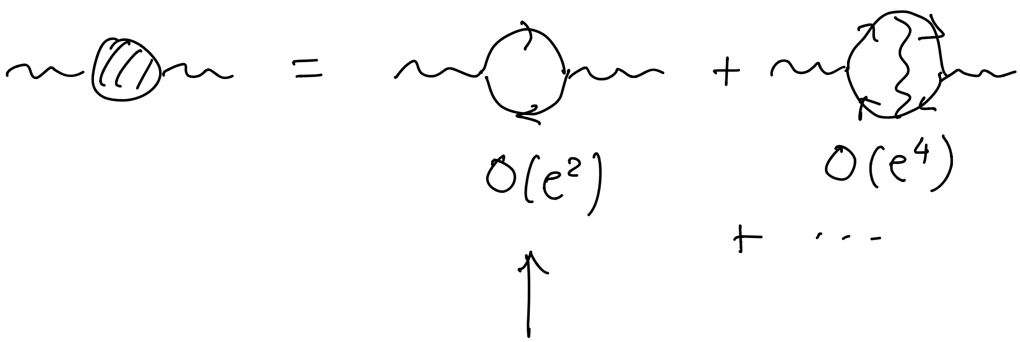
$\hookrightarrow$ 'DRESSED' PHOTON PROPAGATOR

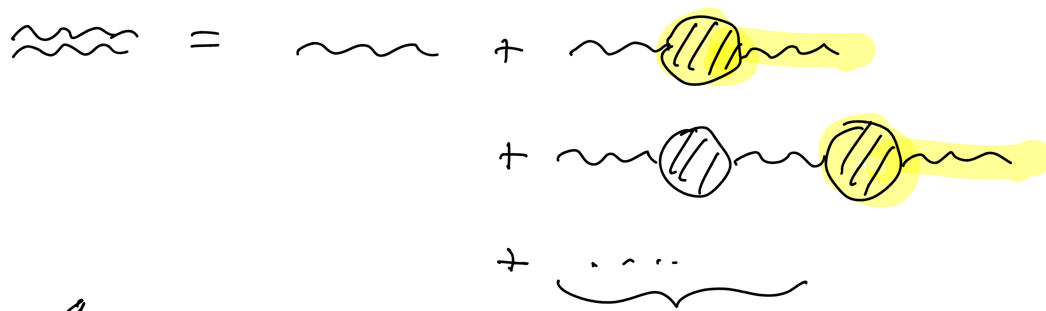


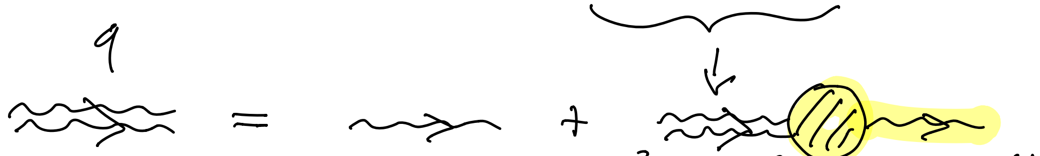
$\sim \quad \frac{e^2}{q^2}$ COULOMB POT

$\longrightarrow$









DYSON
EXPANSION

$$i D^{\mu\nu}(q) = i\,\underbrace{D_0^{\mu\nu}(q)}_{\frac{-g^{\mu\nu}}{q^2+i\varepsilon}} + i D_0^{\mu\alpha}(q)\, i\Pi_{\alpha\beta}(q) \cdot i D^{\beta\nu}(q)$$

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi(q^2)$$

$$= \underbrace{\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}_{} q^2 \Pi(q^2)$$

$$= \Delta^{\mu\nu}(q)$$

$$\Pi_{\alpha\beta}\, D_0^{\beta\nu} = \Delta_{\alpha\beta}\, \cancel{q^2}\, \Pi(q^2)\, \frac{-g^{\beta\nu}}{\cancel{q^2}}$$

$$= -\Delta_\alpha{}^{\nu}\, \Pi(q^2)$$

$$D^{\mu\nu} = D_0^{\mu\nu} - D_0^{\mu\alpha}\, \Pi_{\alpha\beta}\, D_0^{\beta\nu}$$

$$+ D_0^{\mu\alpha}\, \Pi_{\alpha\beta}\, D_0^{\beta\gamma}\, \Pi_{\gamma\delta}\, D_0^{\delta\nu}$$

$$+ \dots$$

$$= D_0^{\mu\nu} + D_0^{\mu\alpha}\, \Delta_\alpha{}^{\nu}\, \Pi$$

$$+ D_0^{\mu\alpha}\, \underbrace{\Delta_\alpha{}^{\gamma}\, \Delta_\gamma{}^{\nu}}_{\Delta_\alpha{}^{\nu}}\, \Pi^2$$

$$+ \dots$$

$$\Delta_\alpha{}^{\gamma} = \left(g_\alpha{}^{\gamma} - \frac{q_\alpha q^\gamma}{q^2} \right)$$

$$\Delta_\alpha{}^{\gamma}\, \Delta_\gamma{}^{\nu} \stackrel{?}{=} \left(g_\alpha{}^{\gamma} - \frac{q_\alpha q^\gamma}{q^2} \right)\left(g_\gamma{}^{\nu} - \frac{q_\gamma q^\nu}{q^2} \right)$$

$$= g_\alpha{}^{\nu} - \frac{q_\alpha q^\gamma}{q^2}$$

$$= \Delta_\alpha{}^{\nu}$$

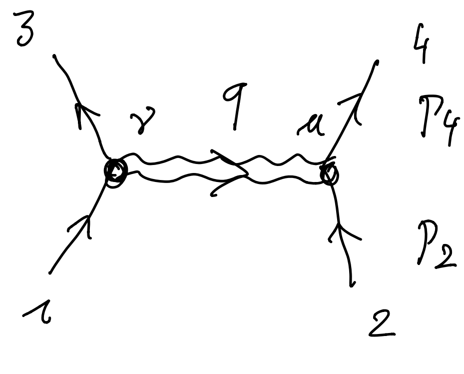
$$D^{\mu\nu} = D_0^{\mu\nu} + D_0^{\mu\alpha}\, \Delta_\alpha{}^{\nu} \left[\Pi + \Pi^2 + \dots \right]$$

$$= -\frac{g^{\mu\nu}}{q^2} - \frac{\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}{q^2}\left[\Pi + \Pi^2 + \dots\right]$$

$$= -\frac{1}{q^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\underbrace{\left[1 + \Pi + \Pi^2 + \dots\right]}_{\frac{1}{1-\Pi(q^2)}} - \frac{q^\mu q^\nu}{q^4}$$

$$\boxed{D^{\mu\nu}(q) = \frac{1}{q^2\left[1-\Pi(q^2)\right]}\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) - \frac{q^\mu q^\nu}{q^4}}$$

$$\frac{e^2}{4\pi} = \alpha_{em} = \frac{1}{137}$$



$$q = p_4 - p_2$$

$$\mathcal{M} = \bar{u}_3 \, (ie\gamma_\nu) \, u_1 \quad \bar{u}_4 \, (ie\gamma_\mu) \, u_2 \cdot i D^{\mu\nu}(q)$$

$$\left\{ \begin{array}{l} q_\nu \, \bar{u}_3 \gamma^\nu u_1 = 0 \\ q_\mu \, \bar{u}_4 \gamma^\mu u_2 = 0 \end{array} \right.$$

$$\hookrightarrow \quad \bar{u}_4 \, (\underbrace{\not{p}_4}_{m} - \underbrace{\not{p}_2}_{m}) \, u_2 = \bar{u}_4 \, (m - m) \, u_1 \stackrel{!}{=} 0$$

$$\boxed{\mathcal{M} = + i e^2 \, \bar{u}_3 \gamma^\mu u_1 \; \bar{u}_4 \gamma_\mu u_2 \cdot \frac{1}{q^2 \left[1 - \Pi(q^2)\right]}}$$

$$V_c(q^2) = \frac{e^2}{q^2}$$

INCLUDING VACUUM POL.

$$V_c(q^2) \longrightarrow V(q^2) = \frac{e^2}{q^2\left[1-\Pi(q^2)\right]}$$

$$\Pi(q^2) = \Pi(0) + q^2\,\Pi'(0) + \cdots$$

$$1-\Pi(q^2) = \left[1-\Pi(0)\right]\left\{1 - q^2\,\frac{\Pi'(0)}{1-\Pi(0)} + O(q^4)\right\}$$

$$V(q^2) = \frac{e^2}{q^2\left[1-\Pi(0)\right]}\left\{1 + q^2\,\frac{\Pi'(0)}{1-\Pi(0)} + O(q^4)\right\}$$

$$\equiv e_R^2 \quad \text{FINITE} \qquad \frac{e_R^2}{4\pi} = \frac{1}{137}$$