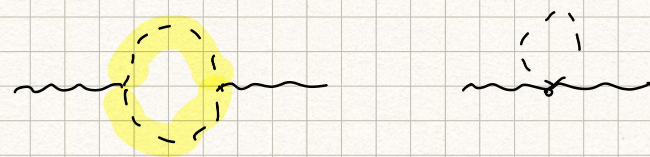
⇒ LECTURE 19 DIMENSIONAL REGULARIZATION

$$\boxed{I_n = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^n} = \underbrace{\frac{i}{(4\pi)^{D/2}}} \frac{(-1)^n}{(\Delta - i\varepsilon)^{n-\frac{D}{2}}} \frac{\Gamma(n - \frac{D}{2})}{\Gamma(n)}}$$

$n = 2$



$$I_2 = \frac{i}{(4\pi)^2} \frac{(4\pi)^{2-D/2}}{\Delta^{2-D/2}} \Gamma(2 - D/2)$$

$\underline{D \to 4}$ $\qquad \varepsilon \equiv 2 - \frac{D}{2}$ $\qquad D < 4$, $\varepsilon > 0$

$$I_2 = \frac{i}{(4\pi)^2} \underbrace{\left(\frac{4\pi}{\Delta}\right)^{\varepsilon}} \underbrace{\Gamma(\varepsilon)} \longleftarrow$$

$$\left(\frac{1}{\varepsilon} - \gamma_E + O(\varepsilon) \right)$$

EULER CONSTANT 0.577...

$$a^{\varepsilon} = e^{\varepsilon \ln a} = 1 + \varepsilon \ln a + \dots$$

$$1 + \varepsilon \ln\left(\frac{4\pi}{\Delta}\right) + O(\varepsilon^2)$$

$$I_2 = \frac{i}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi}{\Delta}\right) + O(\varepsilon) \right\}$$

$$\hookrightarrow \int \frac{d^Dk}{(2\pi)^D} \frac{\text{ODD}\ \#\ k^\mu}{(k^2-\Delta+i\varepsilon)^n} = 0$$

$$\boxed{\hookrightarrow \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2-\Delta+i\varepsilon)^n} = A\, g^{\mu\nu}}$$

MULTIPLY BY $g_{\mu\nu}$

$$\int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^n} = A \underbrace{g_{\mu\nu} g^{\mu\nu}}_{D}$$

$$A = \frac{1}{D} \int \frac{d^Dk}{(2\pi)^D} \left\{ \frac{1}{(k^2-\Delta)^{n-1}} + \frac{\Delta}{(k^2-\Delta)^n} \right\}$$

$$= \frac{1}{D} \frac{i}{(4\pi)^{D/2}} \frac{1}{\Delta^{n-1-D/2}} (-1)^{n-1}$$

$$\cdot \underbrace{\left\{ \frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n-1)} - \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} \right\}}_{\frac{\Gamma(n-1-\frac{D}{2})}{\Gamma(n)} \left\{ \underbrace{(n-1) - (n-1-\frac{D}{2})}_{D/2} \right\}}$$

$$\boxed{= \frac{i}{(4\pi)^{D/2}} \frac{(-1)^{n-1}}{2\,\Gamma(n)} \frac{\Gamma(n-1-D/2)}{\Delta^{n-1-D/2}}}$$

$\underline{n = 2}$

$$= \frac{i}{(4\pi)^2} (4\pi)^{\varepsilon} \underbrace{\frac{(-1)}{2}} \frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}} \quad \Longleftarrow$$

$$\Downarrow \quad \Gamma(-\underbrace{n}+\varepsilon) = \frac{(-1)^n}{n!}\left\{\frac{1}{\varepsilon} + \left(1+\frac{1}{2}+\dots+\frac{1}{n}\right) - \gamma_E + O(\varepsilon)\right\}$$

$n = 1, 2, 3, \dots$

$$= +\frac{i}{(4\pi)^2}\,\frac{\Delta}{2}\left(1+\varepsilon\ln\left(\frac{4\pi}{\Delta}\right)+O(\varepsilon)\right)\cdot\left(\frac{1}{\varepsilon}+1-\gamma_E+O(\varepsilon)\right)$$

$$= \frac{i}{(4\pi)^2}\,\frac{\Delta}{2}\left\{\frac{1}{\varepsilon}+1-\gamma_E+\ln\left(\frac{4\pi}{\Delta}\right)+O(\varepsilon)\right\}$$

$\hookrightarrow$ FIELD DIMENSION IN D-DIM SPACE

$$\mathcal{L}_{\substack{\text{SCALAR}\\ \text{QED}}} = \left(\partial_\mu \phi^\dagger - ie\phi^\dagger A_\mu\right)\left(\partial^\mu \phi + ie\phi A^\mu\right) - m^2 \phi^\dagger \phi - \frac{1}{4}\underbrace{F_{\mu\nu}F^{\mu\nu}}_{(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)}$$

$$\underbrace{S^{(1)} = i\int d^{\cancel{4}\,D}x \; \mathcal{L}^{(1)}_{\text{INT}}}$$

$[S] = 0$ MASS DIMENSION $[\;]$

$\underline{D=4}$	$\underline{D}$
$[d^4x] = -4$	$[d^Dx] = -D$
$[\mathcal{L}] = 4$	$[\mathcal{L}] = D$
$[\phi] = 1$	$[\phi] = \frac{D-2}{2}$
$[A^\mu] = 1$	$[A^\mu] = \frac{D-2}{2}$
$[e] = 0$	$[e] = 2 - \frac{D}{2} = \varepsilon$

$$e \quad \phi^\dagger \quad \partial^\mu \phi \quad A_\mu$$

$$[e] + \frac{D-\cancel{2}}{2} + \cancel{1} + 2\left(\frac{\cancel{D}-2}{2}\right) = \cancel{D}$$

$$[e] = 2 - \frac{D}{2}$$

$$\underset{D=4}{e} \longrightarrow \underset{D\ \text{DIM}}{\underbrace{e}\,\mu^{\varepsilon}}$$

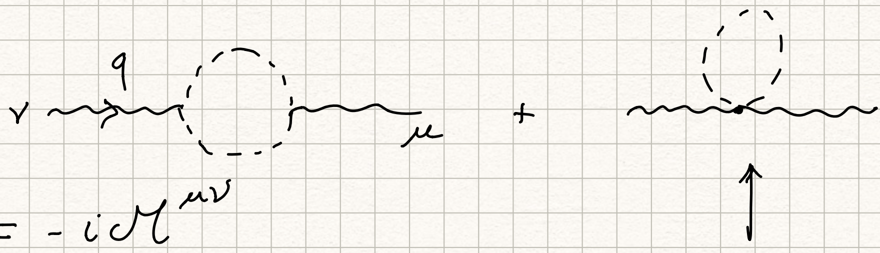
μ: MASS SCALE

$$e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-\Delta)^2}$$

$$\xrightarrow[D\ (\text{DIM. REG.})]{} e^2\,\mu^{2\varepsilon} \underbrace{\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k-\Delta)^2}}$$

$$\left(1 + \varepsilon \ln \mu^2 + O(\varepsilon)\right) \frac{i}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\frac{4\pi}{\Delta} + O(\varepsilon) \right\}$$

$$= \frac{i}{(4\pi)^2} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\left(\frac{\mu^2}{\Delta}\right) + O(\varepsilon) \right\}$$



$$\Pi^{\mu\nu}(q) = -i\mathcal{M}^{\mu\nu}$$

$$= i e^2 \mu^{2\varepsilon} \int_0^1 dx \int \frac{d^{\cancel{4}\,D}k}{(2\pi)^{\cancel{4}\,D}} \frac{1}{[k^2 + \underbrace{q^2 x(1-x) - m^2}_{\hookrightarrow\, -\Delta} + i\varepsilon]^2}$$

$$\cdot \Big\{ -4 k^\mu k^\nu + q^\mu q^\nu [4x(1-x) - 1] + 2 g^{\mu\nu} [k^2 + \underbrace{q^2 (1-x)^2 - m^2}] \Big\}$$

$$\Rightarrow i e^2 \mu^{2\varepsilon} \int_0^1 dx$$

$$\Big\{ \underbrace{-4} \int \frac{d^Dk}{(2\pi)^D} \frac{k^\mu k^\nu}{(k^2-\Delta)^2} \Leftarrow$$

$$+ 2 \underbrace{g^{\mu\nu}} \int \frac{d^Dk}{(2\pi)^D} \frac{k^2}{(k^2-\Delta)^2} \Leftarrow$$

$$+ \Big[2 g^{\mu\nu} (q^2 (1-x)^2 - m^2) + q^\mu q^\nu (4x(1-x) - 1) \Big]$$

$$\cdot \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-\Delta)^2} \Big\} \Leftarrow$$

$$= i e^2 \mu^{2\varepsilon} \frac{i}{(4\pi)^{D/2}} \int_0^1 dx$$

$$\cdot \left\{ \boxed{2 g^{\mu\nu} \frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}} - (1) D g^{\mu\nu} \frac{\Gamma(-1+\varepsilon)}{\Delta^{-1+\varepsilon}}} = 2 g^{\mu\nu} \cdot \frac{\Gamma(\varepsilon)}{\Delta^{-1+\varepsilon}} \right.$$

$$2 g^{\mu\nu} \cdot \underbrace{\left(1 - \frac{D}{2}\right)}_{-1+\varepsilon}$$

$$+ \left[2 g^{\mu\nu} \left(q^2 (1-x)^2 - m^2\right) + q^\mu q^\nu \left(4x(1-x) - 1\right) \right]$$

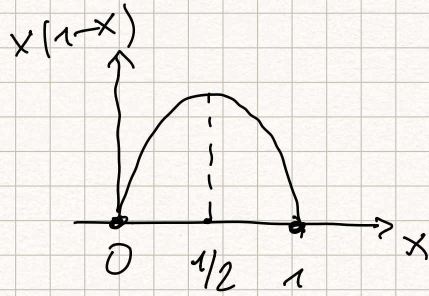
$$\left. \cdot \frac{\Gamma(\varepsilon)}{\Delta^{\varepsilon}} \right\}$$

$$= i e^2 \mu^{2\varepsilon} \frac{i}{(4\pi)^2} \int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^{\varepsilon} \Gamma(\varepsilon)$$

$$\cdot \left\{ 2 g^{\mu\nu} \left(m^2 - q^2 x (1-x) \right) + \left[2 g^{\mu\nu} \left(q^2 (1-x)^2 - m^2\right) + q^\mu q^\nu \left(4x(1-x) - 1\right) \right] \right\}$$

$$\{ \dots \} = 2 g^{\mu\nu} \Big[\underbrace{\cancel{m^2} - q^2 x (1-x) + q^2 (1-x)^2 - \cancel{m^2}}_{q^2 (1-x)(1-2x)} \Big]$$

$$+ \, q^\mu q^\nu \left(2x\,(1-2x) - \underbrace{\cancel{(1-2x)}}_{0} \right)$$



$$\{\ldots\} = 2\,\left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) x\,(1-2x)$$

$$\boxed{\begin{aligned}\Pi^{\mu\nu}(q) = {} & \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \\ & \cdot \left(\frac{-e^2}{(4\pi)^2}\right) 2\Gamma(\varepsilon) \int_0^1 dx \left(\frac{4\pi\mu^2}{m^2 - q^2 x(1-x)}\right)^{\varepsilon} \\ & \cdot x\,(1-2x)\end{aligned}}$$

$$\begin{aligned} = {} & \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right) \cdot \left(\frac{-e^2}{(4\pi)^2}\right) 2 \\ & \cdot \left(\frac{1}{\varepsilon} - \gamma_E + O(\varepsilon) \right) \\ & \cdot \int_0^1 dx \; x\,(1-2x) \left[1 + \varepsilon \ln\left(\frac{4\pi\mu^2}{m^2 - q^2 x(1-x)}\right) \right. \\ & \left. + \, O(\varepsilon) \right] \end{aligned}$$

$$\Pi^{\mu\nu}(q) = \left(q^\mu q^\nu - q^2 g^{\mu\nu}\right)\left(\frac{-e^2}{(4\pi)^2}\right)\cdot 2 \cdot \int_0^1 dx \, x(1-2x) \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{m^2 - q^2 x(1-x)}\right) + O(\varepsilon) \right\}$$

$$Q^2 \equiv -q^2 \gg m^2$$

$$\{\ \} \xrightarrow[Q^2 \gg m^2]{} \left\{ \frac{1}{\varepsilon} - \gamma_E + \ln\left(\frac{4\pi\mu^2}{Q^2}\right) - \ln x(1-x) \right\}$$

$$\Pi^{\mu\nu}(q) \xrightarrow[Q^2 \gg m^2]{} - \frac{e^2}{(4\pi)^2}\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot \frac{1}{3}\left\{ \frac{1}{\varepsilon} - \gamma_E + \ln 4\pi + \ln\frac{\mu^2}{Q^2} + \frac{8}{3} \right\}$$