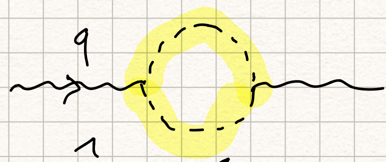
# $\Rightarrow$ LECTURE 18

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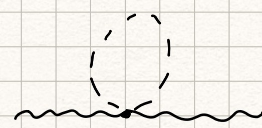
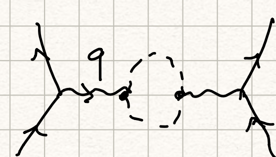


$$\Rightarrow -e_B^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[k^2 + q^2 x(1-x) - m^2 + i\varepsilon\right]^2}$$

$\uparrow$

$$\cdot \Big\{ -4 k^\mu k^\nu + q^\mu q^\nu \Big[ \underbrace{-4x^2 + 4x}_{4x(1-x)} - 1 \Big] + 2 g^{\mu\nu} \left[ k^2 + q^2 (1-x)^2 - m^2 \right] \Big\}$$

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^n}$$




$$\Delta(m^2, q^2, x) = m^2 - q^2 x(1-x)$$

$n = 2 \Rightarrow$ LOGARITHMIC DIV

## REGULARIZATION

1) CUT-OFF METHOD

$$\int^\infty dk \rightarrow \int^\Lambda dk \Rightarrow \ln \Lambda$$

VIOLATES LORENTZ INV.

2) PAULI-VILLARS REGULARIZATION

$$\frac{1}{k^2 - m^2 + i\varepsilon} \longrightarrow \underbrace{\frac{1}{k^2 - m^2 + i\varepsilon}} \underbrace{\left( \frac{m^2 - \Lambda^2}{k^2 - \Lambda^2 + i\varepsilon} \right)}_{\Lambda \to \infty \downarrow \ 1}$$

$k \gg$

$$\frac{1}{k^2} \longrightarrow \frac{1}{k^4}$$

3) $\int d^4 k \longrightarrow \int d^D k$

'T HOOFT

4 DIM $\Rightarrow$ D DIM

$D < 4$

ANALYTIC CONTINUATION

$$\longrightarrow \varepsilon \equiv 2 - \frac{D}{2}$$

$D = 4 \quad : \quad \varepsilon \to 0$

$D < 4 \quad : \quad \varepsilon > 0$

PERFORM LAURENT EXP. AROUND $\varepsilon = 0$

$$I(\varepsilon) = \underbrace{\frac{a_{-1}}{\varepsilon}} + a_0 + a_1 \varepsilon + O(\varepsilon^2)$$

↳ POLE

$$I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^n}$$

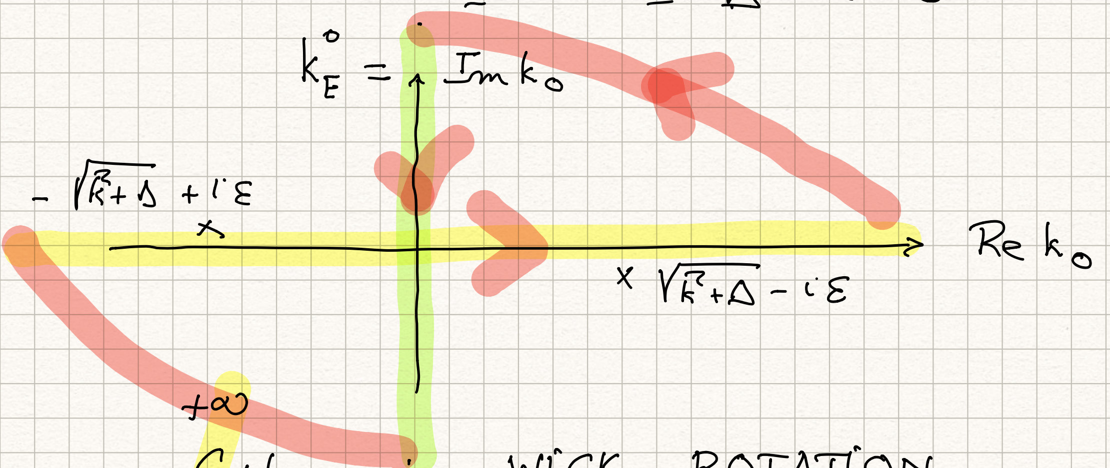
$$\longrightarrow \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^n}$$

$\underline{\Delta > 0}$

$$k^2 - \Delta + i\varepsilon \approx \left(k_0 - \sqrt{\vec{k}^2+\Delta} + i\varepsilon\right) \cdot \left(k_0 + \sqrt{\vec{k}^2+\Delta} - i\varepsilon\right)$$

$$= k_0^2 - (\vec{k}^2 + \Delta) - \cancel{i\varepsilon k_0} + \cancel{i\varepsilon k_0}$$

$$= k^2 - \Delta + i\varepsilon$$




$$\int_{-\infty}^{+\infty} dk_0$$

WICK ROTATION

$$\oint_C dk_0 = 0.$$

$$\int_{-\infty}^{+\infty} dk_0 \cdots + \cancel{\int dk_0} + i \int_{+\infty}^{-\infty} dk_E^0 \cdots = 0$$

## EUCLIDEAN MOMENTA

$$k^\mu = i\, k_E^\mu \qquad\qquad k_E^0 = \mathrm{Im}\, k^0$$

MINKOWSKI $\quad k^\mu(k_0, \bar{k}) \qquad k^2 = (k_0)^2 - \bar{k}^2$

EUCLIDEAN $\quad k_E^\mu(k_E^0, \bar{k}) \qquad k_E^2 \equiv (k_0^E)^2 + \bar{k}^2$

$$= -k_0^2 + \bar{k}^2$$

$$= -k^2$$

$$\underbrace{g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}}$$

$$\int_{-\infty}^{+\infty} dk_0 \frac{1}{(k^2 - \Delta + i\varepsilon)^n} = i \int_{-\infty}^{+\infty} dk_E^0 \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n}$$

$$I = \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - \Delta + i\varepsilon)^n}$$

WICK ROTATION

$$\underbrace{= i \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n}}$$

$\downarrow$ INTRODUCE SPHERICAL COORDINATES IN D DIM. (EUCLIDEAN SPACE)

$\underline{D = 2}$

$$\phi : 0 \to 2\pi$$

$$\int d\Omega_2 = \int_0^{2\pi} d\phi = 2\pi$$

$\underline{D = 3}$

$$\phi : 0 \to 2\pi$$

$$\theta : 0 \to \pi$$

$$d\Omega_3 = d\phi \underbrace{d\theta \sin\theta}_{d\cos\theta}$$

$$\int d\Omega_3 = \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} d\theta \sin\theta}_{2} = 4\pi$$

$\underline{D = 4}$

$$\phi : 0 \to 2\pi$$

$$\theta_1 : 0 \to \pi$$

$$\theta_2 : 0 \to \pi$$

$$d\Omega_4 = \sin^2\theta_1 \, d\theta_1 \, \sin\theta_2 \, d\theta_2 \, d\phi$$

$$\int d\Omega_4 = \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{\pi} d\theta_1 \sin^2\theta_1} \underbrace{\int_0^{\pi} d\theta_2 \sin\theta_2}_{2}$$

$$\downarrow$$

$$x = \cos\theta_1$$

$$2\int_0^1 dx \, (1 - x^2)^{1/2}$$

$$x^2 = u$$

$$2x\,dx = du$$

$$\int_0^1 du\; u^{-1/2}\,(1-u)^{1/2}$$

$$B(n,m) = \int_0^1 dx\; x^{n-1}\,(1-x)^{m-1}$$

$$= \frac{\Gamma(n)\;\Gamma(m)}{\Gamma(n+m)}$$

$$\int d\Omega_4 = 4\pi \cdot B\left(\frac{1}{2},\,\frac{3}{2}\right)$$

$$= 4\pi\;\frac{\Gamma(1/2)\;\Gamma(3/2)}{\Gamma(2)} = 4\pi\;\frac{1}{2}\;\pi = 2\pi^2$$

3-SPHERE

$$\Gamma(n) = (n-1)! \qquad n \text{ INTEGER}$$

$$\Gamma(x) = (x-1)\;\Gamma(x-1)$$

e.g. $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\;\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

# D DIM

$$\phi : 0 \longrightarrow 2\pi$$

$$D-2 \left\{ \begin{array}{l} \theta_1 : 0 \to \pi \\ \vdots \\ \theta_{D-2} : 0 \to \pi \end{array} \right.$$

$$d\Omega_D = d\phi \; \sin\theta_1 \, d\theta_1 \; \sin^2\theta_2 \, d\theta_2 \cdots \sin^{D-2}\theta_{D-2} \, d\theta_{D-2}$$

$$\int_{(D-1)} d\Omega_D = \underbrace{2\pi} \; \underbrace{\int_0^\pi d\theta_1 \sin\theta_1} \; \cdots \; \underbrace{\int_0^\pi d\theta_{D-2} \sin^{D-2}\theta_{D-2}}$$

$$\int_0^\pi d\theta \sin^n\theta = 2\int_0^1 dx \, (1-x^2)^{\frac{n-1}{2}}$$

$$x = \cos\theta$$

$$x^2 = u$$

$$2x\,dx = du$$

$$= \int_0^1 du \; u^{-1/2} (1-u)^{\frac{n-1}{2}}$$

$$= B\left(\frac{1}{2}, \frac{n+1}{2}\right)$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$$

$n = 1, \ldots D-2$

$$\int_{(D-1)} d\Omega_D = 2\pi\,(\pi)^{\frac{D-2}{2}} \frac{\cancel{\Gamma\left(\frac{D-1}{2}\right)}\,\cancel{\Gamma\left(\frac{D-2}{2}\right)}\cancel{\cdots}\,\Gamma(1)}{\Gamma\left(\frac{D}{2}\right)\,\cancel{\Gamma\left(\frac{D-1}{2}\right)}\cancel{\cdots}\,\cancel{\Gamma\left(\frac{3}{2}\right)}}$$

$$= \frac{2\,\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}$$

CHECK

$D = 2 \qquad \frac{2\pi}{\Gamma(1)} = 2\pi$

$D = 3 \qquad \frac{2\pi^{3/2}}{\Gamma\left(\frac{3}{2}\right)} = \frac{2\pi^{3/2}}{\frac{1}{2}\pi^{1/2}} = 4\pi$

$D = 4 \qquad \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$

$$\int_{D-1} d\Omega_D = \frac{2\,\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}$$

$$d^D k_E = d|k_E|\,|k_E|^{D-1}\,d\Omega_D$$

$$I = i \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(-k_E^2 - \Delta + i\varepsilon)^n}$$

$$= i \, \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} \, \frac{(-1)^n}{(2\pi)^D} \int_0^\infty dk_E \; k_E^{D-1} \frac{1}{(k_E^2 + \Delta - i\varepsilon)^n}$$

$$k_E \equiv |k_E|$$

$$\underline{\ell \equiv k_E^2}$$

$$d\ell = 2 k_E \, dk_E$$

$$= i \, \frac{\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} \, \frac{(-1)^n}{(2\pi)^D} \int_{\substack{0 \\ \Downarrow \\ x=1}}^{\infty \Rightarrow x=0} \underbrace{d\ell} \, \frac{\ell^{\frac{D-2}{2}}}{(\ell + \Delta - i\varepsilon)^n}$$

$$dx = -\frac{\Delta}{(\ell+\Delta)^2} d\ell \qquad x = \underbrace{\frac{\Delta}{\ell + \Delta}}$$

$$= -\frac{1}{\Delta} \underbrace{x^2} d\ell \qquad 1 - x = \frac{\ell}{\ell + \Delta}$$

$$d\ell = -\frac{\Delta}{x^2} dx$$

$$\ell^{\frac{D-2}{2}} = (1-x)^{\frac{D-2}{2}} \underbrace{(\ell + \Delta)^{\frac{D-2}{2}}}$$

$$\frac{\ell^{\frac{D-2}{2}}}{(\ell + \Delta - i\varepsilon)^n} = (1-x)^{\frac{D-2}{2}} \left(\frac{x}{\Delta}\right)^{n - \frac{D}{2} + 1}$$

$$\Delta \longrightarrow \Delta - i\varepsilon$$

$$I = \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{\Gamma(D/2)} \cdot \int_0^1 dx \, \frac{x^{n-\frac{D}{2}-1}}{\Delta^{n-\frac{D}{2}}} (1-x)^{\frac{D}{2}-1}$$

$$= \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{\cancel{\Gamma\left(\frac{D}{2}\right)}} \frac{B\left(n-\frac{D}{2}, \frac{D}{2}\right)}{\Delta^{n-\frac{D}{2}}}$$

$$\frac{\Gamma\left(n-\frac{D}{2}\right) \cancel{\Gamma\left(\frac{D}{2}\right)}}{\Gamma(n)}$$

$$\boxed{\begin{aligned} I_n &= \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2-\Delta+i\varepsilon)^n} \\ &= \frac{i}{(4\pi)^{D/2}} \frac{(-1)^n}{(\Delta-i\varepsilon)^{n-\frac{D}{2}}} \frac{\Gamma\left(n-\frac{D}{2}\right)}{\Gamma(n)} \end{aligned}}$$

$\underline{n=2}$

$$I_2 = \frac{i}{(4\pi)^{D/2}} \frac{1}{(\Delta-i\varepsilon)^{2-\frac{D}{2}}} \frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(2)}$$

$$\varepsilon = 2 - \frac{D}{2} \qquad \Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_E + O(\varepsilon)$$

$$D \to 4 \qquad \varepsilon \to 0$$

EULER CONSTANT
0.577...