Theoretical Physics 6a (QFT): SS 2020
Exercise sheet 11

29.06.2020

Exercise 1. (50 points) : β-function in QED

In the lecture notes we have calculated the effective (running) coupling in QED at 1-loop level as:

$$e_R^2(Q^2) = e_R^2 \left\{ 1 + \frac{e_R^2}{12\pi^2} \ln \frac{Q^2}{m^2} + \mathcal{O}(e_R^4) \right\},$$

where e_R^2 on the *rhs* can be interpreted as the coupling defined at $Q^2 = m^2$. The β-function in QED expresses in general how the coupling constant changes with mass scale Q as:

$$\beta(e_R) \equiv Q \frac{de_R}{dQ}.$$

(a)(10 points) Use Eq. (1) to express the β-function at leading order in terms of the renormalized coupling $e_R(Q^2)$.

(b)(10 points) Which power in e_R do the correction terms to the leading order result for the β-function have ?

(c)(10 points) Discuss from the sign of the β function how it behaves at high energies (short distances), corresponding with $Q \to \infty$.

(d)(20 points) By using the result obtained in (a) for the β-function, you obtain a differential equation for $e_R(Q^2)$. Solve this equation by expressing the running coupling $e_R(Q^2)$ at scale Q in terms of the running coupling $e_R(\mu^2)$ at an arbitrary scale μ. Show that when taking $\mu = m$, you find back the result of Eq. (1) up to correction terms of $\mathcal{O}(e_R^6)$.

Exercise 2. (50 points) : 1-loop correction to the propagator in Yukawa theory

Consider the interaction between a scalar field ϕ (with mass M) and a spin 1/2 field ψ (with mass m) described by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}M^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\bar{\psi}\psi\phi,$$

where λ is a coupling constant.

(a)(10 points) Derive the Feynman rule corresponding to the interaction term.

(b)(10 points) The 1-loop correction to the scalar propagator induced by a fermion loop is presented in Fig. 3. Use the Feynman rules to derive the invariant amplitude $\mathcal{M}$ for this diagram, using the momentum labels as indicated on the figure.

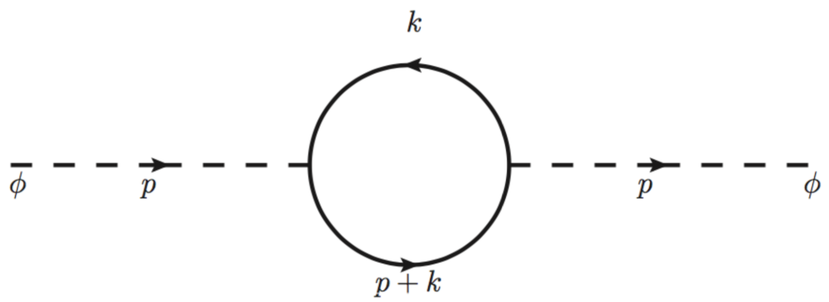

Figure 1: One-loop correction to the scalar propagator in Yukawa theory.

(c)(10 points) Use the Feynman parameterization, and perform the one-loop integral using dimensional regularization (and using the formulas given at the end). Show that the result can be expressed as:

$$\mathcal{M}^{1-loop} = i\frac{4(d-1)\lambda^2\mu^{4-d}}{(4\pi)^{d/2}}\Gamma(1-d/2)\int_0^1 dx \frac{1}{[m^2 - p^2x(1-x)]^{1-d/2}},$$

where d denotes the dimensionality of space-time, and μ is some arbitrary scale to keep the coupling λ dimensionless.

(d)(10 points) The scalar counterterms (CT) that have to be added to the diagram of Fig. 2 correspond with the Feynman rule:

$$\mathcal{M}^{CT} = i\left[p^2\delta_\phi - M^2(\delta_M + \delta_\phi)\right],$$

where δ_ϕ is the counterterm for the field ϕ, and δ_M is the counterterm for the scalar squared mass M^2.

Defining $\varepsilon \equiv 2 - d/2$, expand the above result for the invariant amplitude $\mathcal{M}$ in ε to extract the pole term in $1/\varepsilon$. Use the MS subtraction scheme, i.e. absorb only the divergent parts, and determine the MS expressions for the counterterms δ_ϕ and δ_M.

(e)(10 points) The renormalized propagator of the scalar field is given by

$$\frac{i}{p^2 - M^2 - \Sigma_R(p^2)},$$

with the renormalized self-energy $\Sigma_R(p^2) = i(\mathcal{M}^{1-loop} + \mathcal{M}^{CT})$. Using the above result for the invariant amplitude, what is the expression for $\Sigma_R(p^2)$ in the $\overline{MS}$ scheme? You do not need to perform the Feynman parameter integral. What is the expression for the difference between the pole value (M_P^2) and the $\overline{MS}$ value ($M_{\overline{MS}}^2$) of the squared scalar mass ?

Note: This difference determines the shift in the Higgs mass (M) due to the heavy (mass m) top-quark loop.