

II.

DIRAC FIELD

- 1) DIRAC EQUATION
- 2) QUANTIZATION OF DIRAC FIELD
- 3) FEYNMAN PROPAGATOR OF DIRAC FIELD
- 4) SYMMETRIES OF DIRAC THEORY

# 1) DIRAC EQUATION

## ⇒ DIRAC EQUATION AND SOLUTIONS

↳ DIRAC EQ. : DESCRIBES RELATIVISTIC SPIN 1/2 PARTICLE

$$\boxed{(i \gamma^\mu \partial_\mu - m) \psi(x) = 0}$$

↳ 4x1 DIRAC SPINOR

↳  $\gamma$ -MATRICES : (4x4 MATRICES)

$$\gamma^\mu (\gamma^0, \vec{\gamma})$$

IN DIRAC REPRESENTATION

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\mathbb{1}_{2 \times 2} \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\vec{\sigma} (\sigma^1, \sigma^2, \sigma^3)$$

↳ PAULI MATRICES (2x2)

$\gamma$ -MATRICES SATISFY ANTI-COMMUTATION RELATION:

$$\boxed{[\gamma^\mu, \gamma^\nu]_+ \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbb{1}_{4 \times 4}}$$

↑

ANTI-COMMUTATOR

↳ HERMITICITY CONDITION

$$\gamma^{0\dagger} = \gamma^0$$

$$\gamma^{i\dagger} = -\gamma^i \quad (i=1,2,3)$$

CAN BE COMBINED AS

$$\underline{\underline{\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0}}$$

↳  $\gamma_5$  MATRIX

IN FOLLOWING, WE WILL ALSO ENCOUNTER

A 5-th GAMMA :

$$\underline{\underline{\gamma_5 \equiv \frac{i}{4!} \epsilon_{\mu\nu\alpha\beta} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta}}$$

WITH  $\epsilon_{\mu\nu\alpha\beta}$  TOTAL ANTISYMMETRIC  
LEVI-CIVITA TENSOR IN 4-DIM

CONVENTION USED:

$$\epsilon_{0123} = +1$$

NOTE: THIS CONVENTION IS EQUIVALENT WITH

$$\epsilon^{0123} = -1$$

$$\gamma_5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

$$(\gamma_5)^2 = \mathbb{1}_{4 \times 4}$$

IN DIRAC REPRESENTATION

$$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix} \quad \text{CHANGES UPPER \& LOWER COMPONENTS IN DIRAC SPINOR.}$$

↳ ADJOINT DIRAC FIELD  $\bar{\Psi}$

$$\underline{\underline{\bar{\Psi}(x) \equiv \Psi^\dagger(x) \gamma^0}}$$

TAKE HERMITIAN CONJUGATE OF DIRAC EQ.

$$\left( -i (\partial_\mu \Psi^\dagger) (\gamma^\mu)^\dagger - m \Psi^\dagger \right) = 0$$

MULTIPLY ON RIGHT BY  $\gamma^0$  & USE  $(\gamma^0)^2 = \mathbb{1}$

$$\left( -i (\partial_\mu \Psi^\dagger) \gamma^0 \gamma^\mu (\gamma^0)^2 - m \Psi^\dagger \gamma^0 \right) = 0$$

$$\boxed{\left( i (\partial_\mu \bar{\Psi}) \gamma^\mu + m \bar{\Psi} \right) = 0.}$$

## ↳ PLANE-WAVE SOLUTIONS

CHARACTERIZED BY MOMENTUM  $\vec{p}$ , ENERGY  $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$

§ SPIN PROJECTION  $s = \pm \frac{1}{2}$

(ALONG A CHOSEN QUANTIZATION AXIS)

$$\psi_{\vec{p}, s+}(x) = e^{-i p \cdot x} u(\vec{p}, s) \quad \begin{array}{l} \text{POSITIVE ENERGY} \\ \text{SOLUTION} \\ \text{(PARTICLE)} \end{array}$$

$$\psi_{\vec{p}, s-}(x) = e^{+i p \cdot x} v(\vec{p}, s) \quad \begin{array}{l} \text{NEGATIVE ENERGY} \\ \text{SOLUTION} \\ \text{(ANTI-PARTICLE)} \end{array}$$

↳ PLUG INTO DIRAC EQ.

$$\left\{ \begin{array}{l} (\not{p} - m) u(\vec{p}, s) = 0 \\ (\not{p} + m) v(\vec{p}, s) = 0 \end{array} \right.$$

WITH  $\not{p} \equiv \gamma_{\mu} p^{\mu}$

EQUIVALENTLY FOR ADJOINT SPINORS

$$\begin{array}{l} \bar{u}(\vec{p}, s) \equiv u^{\dagger}(\vec{p}, s) \gamma^0 \Rightarrow \left\{ \begin{array}{l} \bar{u}(\vec{p}, s) (\not{p} - m) = 0 \\ \bar{v}(\vec{p}, s) (\not{p} + m) = 0 \end{array} \right. \end{array}$$

↳ FOR SPIN QUANTIZED ALONG AXIS  $\hat{m}$

$$\bar{S} = \frac{\hbar \sigma}{2} \quad \leadsto \quad \underline{\underline{\left( \frac{\sigma}{2} \cdot \hat{m} \right) \chi_{\uparrow} = s \chi_{\uparrow}}}$$

$$s = \pm \frac{1}{2}$$

$\chi_{\uparrow}$  : 2x1 PAULI-SPINOR

- $\hat{m} = \bar{e}_z$       SUPERSCRIPT ON  $\chi$  INDICATES QUANTIZATION AXIS

$$\chi_{+\frac{1}{2}}^{(z)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SPIN UP

$$\chi_{-\frac{1}{2}}^{(z)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

SPIN DOWN

- $\hat{m} = \bar{e}_x$

$$\sigma_x \chi_{\uparrow}^{(x)} = (2s) \chi_{\uparrow}^{(x)}$$

e.g.  $s = +\frac{1}{2}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow a = b$$

$$\chi_{+\frac{1}{2}}^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\chi_{-\frac{1}{2}}^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

↑  
NORMALIZATION  $\chi_{\uparrow}^{\dagger} \chi_{\uparrow} = \delta_{ss'}$

↳ IN DIRAC REPRESENTATION : SOLUTION U

$$\left( \gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} - m \right) U(\vec{p}, s) = 0$$

$$\downarrow \quad U(\vec{p}, s) = N \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix}$$

↑  
NORMALIZATION

$$\begin{pmatrix} E_{\vec{p}} - m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & E_{\vec{p}} + m \end{pmatrix} \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix} = 0$$

$$\hookrightarrow (E_{\vec{p}} + m) \varphi_s = (\vec{\sigma} \cdot \vec{p}) \chi_s$$

$$\therefore \left\| U(\vec{p}, s) = N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi_s \end{pmatrix} \right.$$

WHERE  $\chi_s$  IS A PAULI-SPINOR  $s = \pm \frac{1}{2}$

NORMALIZATION (COVARIANT NORMALIZATION)

$$\left\| \begin{aligned} \bar{U}(\vec{p}, s) U(\vec{p}, s') &= (2m) \delta_{ss'} \\ U^\dagger(\vec{p}, s) U(\vec{p}, s') &= (2E_{\vec{p}}) \delta_{ss'} \end{aligned} \right.$$

$$\bar{U}(\bar{P}, s) U(\bar{P}, s') = N^2 \left( \chi_s^+ \quad -\chi_s^+ \frac{\bar{\sigma} \cdot \bar{P}}{E_{\bar{P}} + m} \right) \begin{pmatrix} \chi_{s'} \\ \frac{\bar{\sigma} \cdot \bar{P}}{E_{\bar{P}} + m} \chi_{s'} \end{pmatrix}$$

$$= N^2 \left\{ \chi_s^+ \chi_{s'} - \chi_s^+ \frac{(\bar{\sigma} \cdot \bar{P})(\bar{\sigma} \cdot \bar{P})}{(E_{\bar{P}} + m)^2} \chi_{s'} \right\}$$

$$\downarrow (\bar{\sigma} \cdot \bar{P})(\bar{\sigma} \cdot \bar{P}) = \bar{P}^2 = E_{\bar{P}}^2 - m^2$$

$$= N^2 \chi_s^+ \chi_{s'} \left\{ 1 - \frac{E_{\bar{P}}^2 - m^2}{(E_{\bar{P}} + m)^2} \right\}$$

$$\downarrow$$

$$= N^2 \delta_{ss'} \frac{(2m)}{E_{\bar{P}} + m}$$

BY REQUIRING  $\bar{U}(\bar{P}, s) U(\bar{P}, s') = (2m) \delta_{ss'}$ ,

$$\underline{\underline{N = \sqrt{E_{\bar{P}} + m}}}$$

↳ WITH THIS CONVENTION ONE SHOWS THAT

$$U^{\dagger}(\bar{P}, s) U(\bar{P}, s') = 2E_{\bar{P}} \delta_{ss'}$$



↳ IN DIRAC REPRESENTATION: SOLUTION  $\psi$

$$\left( \gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} + m \right) \psi(\vec{p}, s) = 0.$$

$$\downarrow \quad \psi(\vec{p}, s) = N \begin{pmatrix} \varphi'_s \\ \chi'_s \end{pmatrix}$$

$$\begin{pmatrix} E_{\vec{p}} + m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -(E_{\vec{p}} - m) \end{pmatrix} \begin{pmatrix} \varphi'_s \\ \chi'_s \end{pmatrix} = 0$$

$$\downarrow \quad (E_{\vec{p}} + m) \varphi'_s - \vec{\sigma} \cdot \vec{p} \chi'_s = 0$$

$$\therefore \left\| \psi(\vec{p}, s) = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi'_s \\ \chi'_s \end{pmatrix} \right.$$

$$\text{WITH } \left\| \begin{aligned} \chi'_{+\frac{1}{2}} &= \chi_{-\frac{1}{2}} \\ \chi'_{-\frac{1}{2}} &= \chi_{+\frac{1}{2}} \end{aligned} \right.$$

DIRAC: ABSENCE OF NEG. ENERGY SOLUTION  
OF SPIN PROJ  $-\frac{1}{2}$   
INTERPRETED AS ANTI-PARTICLE WITH SPIN  
PROJ  $+\frac{1}{2}$

## NORMALIZATION

$$\begin{aligned}
& \bar{v}(\bar{p}, s) v(\bar{p}, s') \\
&= N^2 \left( \chi_{s'}^{i\dagger} \frac{\bar{\sigma} \cdot \bar{p}}{E_{\bar{p}} + m} - \chi_{s'}^{i\dagger} \right) \begin{pmatrix} \frac{\bar{\sigma} \cdot \bar{p}}{E_{\bar{p}} + m} \chi_{s'}^i \\ \chi_{s'}^i \end{pmatrix} \\
&= N^2 \left\{ \chi_{s'}^{i\dagger} \frac{(\bar{\sigma} \cdot \bar{p})(\bar{\sigma} \cdot \bar{p})}{(E_{\bar{p}} + m)^2} \chi_{s'}^i - \chi_{s'}^{i\dagger} \chi_{s'}^i \right\} \\
&= N^2 \chi_{s'}^{i\dagger} \chi_{s'}^i \left\{ \frac{E_{\bar{p}}^2 - m^2}{(E_{\bar{p}} + m)^2} - 1 \right\} \\
&\quad \downarrow \\
&= N^2 \delta_{ss'} \frac{(-2m)}{E_{\bar{p}} + m}
\end{aligned}$$

$$\downarrow \text{ WITH CHOICE } N = \sqrt{E_{\bar{p}} + m}$$

$$= -(2m) \delta_{ss'}$$

$$\begin{aligned}
\therefore \quad & \left\| \begin{aligned} \bar{v}(\bar{p}, s) v(\bar{p}, s') &= -(2m) \delta_{ss'} \\ v^\dagger(\bar{p}, s) v(\bar{p}, s') &= (2E_{\bar{p}}) \delta_{ss'} \end{aligned} \right.
\end{aligned}$$

↳ WE CAN ALSO EASILY CHECK THE ORTHOGONALITY OF PARTICLE ( $u$ ) AND ANTI-PARTICLE ( $v$ ) SPINORS:

$$\begin{cases} \bar{u}(\vec{p}, s) v(\vec{p}, s') = 0 \\ \bar{v}(\vec{p}, s) u(\vec{p}, s') = 0 \end{cases}$$

NOTE: BECAUSE WE WANT TO WORK WITH LORENTZ INVARIANT NORMALIZATION WE NORMALIZE

$$\bar{u}(\vec{p}, s) u(\vec{p}, s') \text{ TO A CONSTANT } (2m) \delta_{ss'}$$

IN CONTRAST:

$$u^\dagger(\vec{p}, s) u(\vec{p}, s') = 2E_{\vec{p}} \delta_{ss'}$$

IS NOT LORENTZ INVARIANT AS IT INVOLVES THE ENERGY  $E_{\vec{p}}$

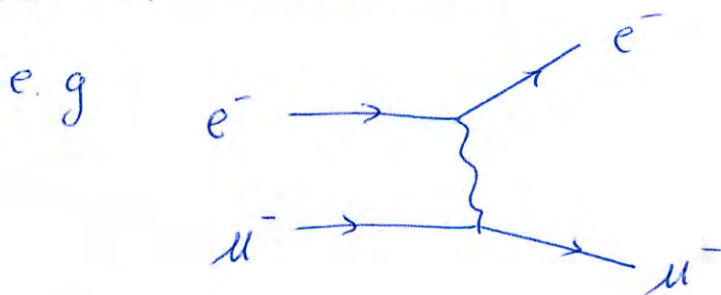
∴ IN QFT FOR DIRAC PARTICLES:

WE WILL WORK WITH  $\psi(x)$  AND  $\bar{\psi}(x)$

( $\bar{\psi}$  INSTEAD OF  $\psi^\dagger$ )

## ⇒ SPIN SUMS

IN FOLLOWING WE WILL FREQUENTLY ENCOUNTER PROCESSES IN WHICH THE POLARIZATION OF INITIAL SPIN  $1/2$  BEAM PARTICLE IS RANDOMLY ORIENTED, AND POLARIZATION OF FINAL SPIN  $1/2$  PARTICLE IS NOT OBSERVED



$$\text{INITIAL } e^- : \text{SPIN } s_e \Rightarrow \frac{1}{2} \sum_{s_e}$$

$$\text{FINAL } e^- : \text{SPIN } s'_e \Rightarrow \sum_{s'_e}$$

SUCH UNPOLARIZED PROCESSES WILL LEAD TO SPIN SUMS OF FORM

$$\sum_{s=\pm\frac{1}{2}} U(\vec{p}, s) \bar{U}(\vec{p}, s)$$

BY PLUGGING IN A SOLUTION

$$(E_{\vec{p}} + m) \sum_s \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \chi_s \end{pmatrix} \left( \chi_s^\dagger - \chi_s^\dagger \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \right)$$

$$= (E_{\vec{p}} + m) \cdot \begin{pmatrix} \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} & - \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} & - \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \left( \sum_{\lambda} \chi_{\lambda} \chi_{\lambda}^{\dagger} \right) \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \end{pmatrix}$$

↓ USING COMPLETENESS FOR PAULI SPINORS

$$\sum_{\lambda = \pm \frac{1}{2}} \chi_{\lambda} \chi_{\lambda}^{\dagger} = \mathbb{1}_{2 \times 2}$$

$$= (E_{\vec{p}} + m) \begin{pmatrix} \mathbb{1} & - \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}} + m} & - \frac{\vec{p}^2}{(E_{\vec{p}} + m)^2} \end{pmatrix}$$

$$= \begin{pmatrix} E_{\vec{p}} + m & - \vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & - (E_{\vec{p}} - m) \end{pmatrix}$$

∴

$$\sum_{\lambda = \pm \frac{1}{2}} u(\vec{p}, \lambda) \bar{u}(\vec{p}, \lambda) = \not{p} + m.$$

AND ANALOGOUSLY

$$\sum_{\lambda = \pm \frac{1}{2}} v(\vec{p}, \lambda) \bar{v}(\vec{p}, \lambda) = \not{p} - m.$$

## ⇒ DIRAC LAGRANGIAN

↳ INTERPRETING  $\Psi(x)$  AND  $\bar{\Psi}(x)$

AS INDEPENDENT FIELDS SATISFYING

$$\left\| \begin{aligned} (i \gamma^\mu \partial_\mu - m) \Psi &= 0. \end{aligned} \right.$$

$$\left\| \begin{aligned} (i (\partial_\mu \bar{\Psi}) \gamma^\mu + m \bar{\Psi}) &= 0. \end{aligned} \right.$$

WE CAN DERIVE DIRAC EQ.

FROM THE DIRAC LAGRANGIAN:

$$\boxed{\mathcal{L}_{\text{DIRAC}} = \bar{\Psi}(x) (i \gamma^\mu \partial_\mu - m) \Psi(x)}$$

↳ EULER LAGRANGE EQS.

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\Psi}} &= (i \gamma^\mu \partial_\mu - m) \Psi \\ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\Psi})} &= 0 \end{aligned} \right\} \Rightarrow (i \gamma^\mu \partial_\mu - m) \Psi = 0$$

$$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial \Psi} &= \bar{\Psi} (-m) \\ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} &= \bar{\Psi} i \gamma^\mu \end{aligned} \right\} \Rightarrow i (\partial_\mu \bar{\Psi}) \gamma^\mu + m \bar{\Psi} = 0$$

## ⇒ DIRAC HAMILTONIAN

↳ CONJUGATE MOMENTA

$$\psi \rightarrow \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i \bar{\psi} \gamma^0 = i \psi^\dagger$$

$$\bar{\psi} \rightarrow \bar{\pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

↳ HAMILTONIAN

$$\mathcal{H} = \pi \dot{\psi} + \cancel{\bar{\psi} \dot{\bar{\pi}}} - \mathcal{L}$$

$$= i \bar{\psi} \gamma^0 \partial_0 \psi - \bar{\psi} (i \gamma^u \partial_u - m) \psi$$

$$= \bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi$$

WITH  $\vec{\nabla}^i = \frac{\partial}{\partial x^i} = \partial_i$

$$H = \int d^3 \vec{x} \bar{\psi}(x) (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi(x)$$

## ⇒ ENERGY - MOMENTUM TENSOR

$$\hookrightarrow T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} (\partial^\nu \psi) - g^{\mu\nu} \mathcal{L}$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$T_{\text{DIRAC}}^{\mu\nu} = \bar{\psi} (i \gamma^\mu \partial^\nu \psi - g^{\mu\nu} \bar{\psi} (i \gamma^\alpha \partial_\alpha - m) \psi)$$

↳ CONSERVED QUANTITY

4 - MOMENTUM  $P^\nu$

$$P^\nu = \int d^3 \bar{x} T^{0\nu}$$

$$= \int d^3 \bar{x} \left( \pi \partial^\nu \psi - g^{0\nu} \bar{\psi} (i \gamma^\mu \partial_\mu - m) \psi \right)$$

$$P^0 = H$$

$$P^i = \int d^3 \bar{x} \pi (\partial^i \psi)$$

( $i=1, 2, 3$ )

$$= \int d^3 \bar{x} \psi^\dagger (-i \vec{\nabla}^i) \psi$$

→ 3 - MOMENTUM  
OF DIRAC FIELD



## 2) QUANTIZATION OF DIRAC FIELD

⇒ FERMI-DIRAC STATISTICS : ANTI-COMMUTATION  
RELATIONS

↳ NORMAL MODE EXPANSION FOR  $\Psi$

$$\Psi(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ a(\vec{p}, s) u(\vec{p}, s) e^{-ip \cdot x} + b^\dagger(\vec{p}, s) v(\vec{p}, s) e^{+ip \cdot x} \right\}$$

$$(i\gamma^\mu \partial_\mu - m) u(\vec{p}, s) e^{-ip \cdot x} = (\not{p} - m) u(\vec{p}, s) e^{-ip \cdot x} = 0$$

$$(i\gamma^\mu \partial_\mu - m) v(\vec{p}, s) e^{+ip \cdot x} = -(\not{p} + m) v(\vec{p}, s) e^{+ip \cdot x} = 0$$

↳ NORMAL MODE EXPANSION FOR ADJOINT FIELD  $\bar{\Psi}$

$$\bar{\Psi}(x) = \sum_{s=\pm\frac{1}{2}} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ a^\dagger(\vec{p}, s) \bar{u}(\vec{p}, s) e^{+ip \cdot x} + b(\vec{p}, s) \bar{v}(\vec{p}, s) e^{-ip \cdot x} \right\}$$

↳ SINGLE-PARTICLE STATE ( $e^-$ : ELECTRON OF MOMENTUM  $\vec{p}$ , SPIN PROJ  $s$ )

$$|\vec{p}, s\rangle_{e^-} \equiv \sqrt{2E_{\vec{p}}} a^+(\vec{p}, s) |0\rangle$$

↳ SINGLE ANTI-PARTICLE STATE ( $e^+$ : POSITRON OF MOMENTUM  $\vec{p}$ , SPIN PROJ  $s$ )

$$|\vec{p}, s\rangle_{e^+} \equiv \sqrt{2E_{\vec{p}}} b^+(\vec{p}, s) |0\rangle$$

↳ VACUUM

$$a(\vec{p}, s) |0\rangle = 0$$

$$b(\vec{p}, s) |0\rangle = 0$$

↳ FERMI-DIRAC STATISTICS.

$e^-$  AND  $e^+$  ARE BOTH SPIN  $1/2$  FERMIONS.

↳ PAULI PRINCIPLE:

2 FERMIONS CANNOT OCCUPY THE SAME STATE, STATE IS ANTI-SYMMETRIC UNDER EXCHANGE OF 2 FERMIONS

$$a^+(\vec{p}, s) a^+(\vec{p}, s) |0\rangle = 0$$

$$b^+(\vec{p}, s) b^+(\vec{p}, s) |0\rangle = 0$$

$$a^+(\vec{p}, s) a^+(\vec{p}', s') |0\rangle = -a^+(\vec{p}', s') a^+(\vec{p}, s) |0\rangle$$

$$b^+(\vec{p}, s) b^+(\vec{p}', s') |0\rangle = -b^+(\vec{p}', s') b^+(\vec{p}, s) |0\rangle$$

CAN BE SATISFIED BY

IMPOSING ANTI-COMMUTATION RELATIONS:

$$[a(\bar{p}, s), a(\bar{p}', s')]_+ = 0$$

$$[a^+(\bar{p}, s), a^+(\bar{p}', s')]_+ = 0$$

$$[a(\bar{p}, s), a^+(\bar{p}', s')]_+ = (2\pi)^3 \delta^3(\bar{p} - \bar{p}') \delta_{ss'}$$

$$[b(\bar{p}, s), b(\bar{p}', s')]_+ = 0$$

$$[b^+(\bar{p}, s), b^+(\bar{p}', s')]_+ = 0$$

$$[b(\bar{p}, s), b^+(\bar{p}', s')]_+ = (2\pi)^3 \delta^3(\bar{p} - \bar{p}') \delta_{ss'}$$

§ ALL ANTI-COMMUTATORS BETWEEN  
PARTICLES & ANTI-PARTICLES VANISH

$$[a(\bar{p}, s), b(\bar{p}', s')]_+ = [a(\bar{p}, s), b^+(\bar{p}', s')]_+ = 0$$

$$[a^+(\bar{p}, s), b(\bar{p}', s')]_+ = [a^+(\bar{p}, s), b^+(\bar{p}', s')]_+ = 0$$

WITH ANTI-COMMUTATOR

$$[A, B]_+ \equiv AB + BA$$

OFTEN ONE ALSO USES SYMBOL  $\{A, B\}$  FOR ANTI-COMMUTATOR.

↳ WITH ANTI-COMMUTATION RELATIONS :  
SINGLE-PARTICLE STATES ARE NORMALIZED AS :

$$e^{-} \langle \bar{p}', s' | \bar{p}, s \rangle_{e^{-}} = \sqrt{2E_{\bar{p}'}} \sqrt{2E_{\bar{p}}} \langle 0 | a(\bar{p}', s') a^{\dagger}(\bar{p}, s) | 0 \rangle$$

$$= \sqrt{2E_{\bar{p}'}} \sqrt{2E_{\bar{p}}} \langle 0 | [a(\bar{p}', s'), a^{\dagger}(\bar{p}, s)]_{+} | 0 \rangle$$

$$\underline{e^{-} \langle \bar{p}', s' | \bar{p}, s \rangle_{e^{-}} = (2\pi)^3 2E_{\bar{p}} \delta^3(\bar{p} - \bar{p}') \delta_{ss'}}$$

AND ANALOGOUSLY

$$e^{+} \langle \bar{p}', s' | \bar{p}, s \rangle_{e^{+}} = (2\pi)^3 2E_{\bar{p}} \delta^3(\bar{p} - \bar{p}') \delta_{ss'}$$

⇒ ANTI-COMMUTATION RELATIONS FOR FIELDS

↳  $\psi(\bar{x}, t) \rightarrow \pi_{\psi}(\bar{x}, t) = i \dot{\psi}^{\dagger}(\bar{x}, t)$   
CONJUGATE MOMENTUM FIELD

EQUAL TIME ANTI-COMMUTATOR

$$[\psi(\bar{x}, t), \psi^{\dagger}(\bar{x}', t)]_{+}$$

$$= \sum_s \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}}$$

$$\left[ \begin{aligned} & a(\bar{p}, s) u(\bar{p}, s) e^{-iE_{\bar{p}}t} e^{i\bar{p} \cdot \bar{x}} + b^{\dagger}(\bar{p}, s) v(\bar{p}, s) e^{+iE_{\bar{p}}t} e^{-i\bar{p} \cdot \bar{x}} \\ & a^{\dagger}(\bar{p}', s') u^{\dagger}(\bar{p}', s') e^{+iE_{\bar{p}'}t} e^{-i\bar{p}' \cdot \bar{x}'} + b(\bar{p}', s') v^{\dagger}(\bar{p}', s') e^{-iE_{\bar{p}'}t} e^{+i\bar{p}' \cdot \bar{x}'} \end{aligned} \right]_{+}$$

$$\begin{aligned}
 & \downarrow \quad [a(\vec{p}, s), a^\dagger(\vec{p}', s')]_+ \dots \\
 & = [b(\vec{p}, s), b^\dagger(\vec{p}', s')]_+ \\
 & = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \delta_{ss'}
 \end{aligned}$$

$$\begin{aligned}
 & [\psi(\vec{x}, t), \psi^\dagger(\vec{x}', t)]_+ \\
 & = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left\{ u(\vec{p}, s) u^\dagger(\vec{p}, s) e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right. \\
 & \quad \left. + v(\vec{p}, s) v^\dagger(\vec{p}, s) e^{-i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & \downarrow \quad \sum_s u(\vec{p}, s) u^\dagger(\vec{p}, s) = (\not{p} + m) \gamma^0 \\
 & \quad \sum_s v(\vec{p}, s) v^\dagger(\vec{p}, s) = (\not{p} - m) \gamma^0
 \end{aligned}$$

$$\begin{aligned}
 & = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left\{ (\gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} + m) \gamma^0 e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right. \\
 & \quad \left. + (\gamma^0 E_{\vec{p}} - \vec{\gamma} \cdot \vec{p} - m) \gamma^0 e^{-i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\}
 \end{aligned}$$

$$\downarrow \quad \text{IN } 2^{\text{O}} \text{ TERM} \quad \vec{p} \rightarrow -\vec{p}$$

$$\begin{aligned}
 & = \int \frac{d^3 \vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left\{ 2E_{\vec{p}} e^{i\vec{p} \cdot (\vec{x} - \vec{x}')} \right\} \mathbb{1}_{4 \times 4} \\
 & \quad \hookrightarrow 4 \times 4 \text{ MATRIX IN DIRAC SPACE} \\
 & = \delta^3(\vec{x} - \vec{x}') \mathbb{1}_{4 \times 4}
 \end{aligned}$$

$$\circ \circ \quad \left\| \begin{aligned} [\Psi(\bar{x}, t), \Psi^\dagger(\bar{x}', t)]_+ &= \delta^3(\bar{x} - \bar{x}') \mathbb{1}_{4 \times 4} \\ [\Psi(\bar{x}, t), \Psi(\bar{x}', t)]_+ &= 0 \\ [\Psi^\dagger(\bar{x}, t), \Psi^\dagger(\bar{x}', t)]_+ &= 0 \end{aligned} \right.$$

⇒ HAMILTONIAN IN NORMAL MODE EXPANSION

$$\hookrightarrow H = \int d^3\bar{x} \bar{\Psi}(x) (-i \vec{\gamma} \cdot \vec{\nabla} + m) \Psi(x)$$

↓ INSERT NORMAL MODE EXPANSIONS  
FOR  $\Psi(x)$  &  $\bar{\Psi}(x)$

$$H = \int d^3\bar{x} \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}}$$

$$\cdot \left( a^\dagger(\bar{p}, s) \bar{u}(\bar{p}, s) e^{+i\bar{p} \cdot x} + b(\bar{p}, s) \bar{v}(\bar{p}, s) e^{-i\bar{p} \cdot x} \right)$$

$$\cdot (-i \vec{\gamma} \cdot \vec{\nabla} + m) \left( a(\bar{p}', s') u(\bar{p}', s') e^{-i\bar{p}' \cdot x} + b^\dagger(\bar{p}', s') v(\bar{p}', s') e^{+i\bar{p}' \cdot x} \right)$$

$$H = \int d^3\bar{x} \sum_s \sum_{s'} \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3\bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}}$$

$$\cdot \left( a^\dagger(\bar{p}, s) \bar{u}(\bar{p}, s) e^{+i\bar{p} \cdot x} + b(\bar{p}, s) \bar{v}(\bar{p}, s) e^{-i\bar{p} \cdot x} \right)$$

$$\cdot \left( a(\bar{p}', s') (\vec{\gamma} \cdot \bar{p}' + m) u(\bar{p}', s') e^{-i\bar{p}' \cdot x} + b^\dagger(\bar{p}', s') (-\vec{\gamma} \cdot \bar{p}' + m) v(\bar{p}', s') e^{+i\bar{p}' \cdot x} \right)$$

$$\text{USING: } (\not{p}' - m) U(\bar{p}', s') = 0$$

$$\Downarrow$$

$$(\gamma^0 E_{\bar{p}'} - \vec{\gamma} \cdot \vec{p}' - m) U(\bar{p}', s') = 0.$$

$$(\not{p}' + m) v(\bar{p}', s') = 0$$

$$\Downarrow$$

$$(\gamma^0 E_{\bar{p}'} - \vec{\gamma} \cdot \vec{p}' + m) v(\bar{p}', s') = 0$$

$$H = \sum_{s} \sum_{s'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} \int d^3 \vec{x} \left\{ \begin{aligned} & e^{i(E_{\vec{p}} - E_{\vec{p}'})t - i(\vec{p} - \vec{p}') \cdot \vec{x}} a^+(\vec{p}, s) a(\vec{p}', s') \\ & \cdot \bar{u}(\vec{p}, s) \gamma^0 E_{\vec{p}'} v(\vec{p}', s') \\ - & e^{i(E_{\vec{p}} + E_{\vec{p}'})t - i(\vec{p} + \vec{p}') \cdot \vec{x}} a^+(\vec{p}, s) b^+(\vec{p}', s') \\ & \cdot \bar{u}(\vec{p}, s) \gamma^0 E_{\vec{p}'} v(\vec{p}', s') \\ + & e^{-i(E_{\vec{p}} + E_{\vec{p}'})t + i(\vec{p} + \vec{p}') \cdot \vec{x}} b(\vec{p}, s) a(\vec{p}', s') \\ & \cdot \bar{v}(\vec{p}, s) \gamma^0 E_{\vec{p}'} u(\vec{p}', s') \\ - & e^{-i(E_{\vec{p}} - E_{\vec{p}'})t + i(\vec{p} - \vec{p}') \cdot \vec{x}} b(\vec{p}, s) b^+(\vec{p}', s') \\ & \cdot \bar{v}(\vec{p}, s) \gamma^0 E_{\vec{p}'} v(\vec{p}', s') \end{aligned} \right\}$$

$$\downarrow \int d^3 \bar{x} e^{-i(\bar{p} - \bar{p}') \cdot \bar{x}} = (2\pi)^3 \delta^3(\bar{p} - \bar{p}')$$

$$H = \sum_{\lambda} \sum_{\lambda'} \int \frac{d^3 \bar{p}}{(2\pi)^3} \frac{E_{\bar{p}}}{2E_{\bar{p}}} \cdot \left\{ \begin{aligned} & a^{\dagger}(\bar{p}, \lambda) a(\bar{p}, \lambda') U^{\dagger}(\bar{p}, \lambda) U(\bar{p}, \lambda') \\ & - e^{2iE_{\bar{p}}t} a^{\dagger}(\bar{p}, \lambda) b^{\dagger}(-\bar{p}, \lambda') U^{\dagger}(\bar{p}, \lambda) \mathcal{V}(-\bar{p}, \lambda') \\ & + e^{-2iE_{\bar{p}}t} b(\bar{p}, \lambda) a(-\bar{p}, \lambda') \mathcal{V}^{\dagger}(\bar{p}, \lambda) U(-\bar{p}, \lambda') \\ & - b(\bar{p}, \lambda) b^{\dagger}(\bar{p}, \lambda') \mathcal{V}^{\dagger}(\bar{p}, \lambda) \mathcal{V}(\bar{p}, \lambda') \end{aligned} \right\}$$

ORTHOGONALITY

$$\downarrow U^{\dagger}(\bar{p}, \lambda) U(\bar{p}, \lambda') = \delta_{\lambda\lambda'} (2E_{\bar{p}})$$

$$\mathcal{V}^{\dagger}(\bar{p}, \lambda) \mathcal{V}(\bar{p}, \lambda') = \delta_{\lambda\lambda'} (2E_{\bar{p}})$$

$$U^{\dagger}(\bar{p}, \lambda) \mathcal{V}(-\bar{p}, \lambda') = 0$$

$$\mathcal{V}^{\dagger}(\bar{p}, \lambda) U(-\bar{p}, \lambda') = 0$$

$$\| H = \sum_{\lambda} \int \frac{d^3 \bar{p}}{(2\pi)^3} E_{\bar{p}} \left\{ \begin{aligned} & a^{\dagger}(\bar{p}, \lambda) a(\bar{p}, \lambda) \\ & - b(\bar{p}, \lambda) b^{\dagger}(\bar{p}, \lambda) \end{aligned} \right\}$$



USING  $[b(\vec{p}, \lambda), b^\dagger(\vec{p}, \lambda)]_+ = (2\pi)^3 \delta^3(0)$

$$H = \sum_{\lambda} \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left\{ a^\dagger(\vec{p}, \lambda) a(\vec{p}, \lambda) + b^\dagger(\vec{p}, \lambda) b(\vec{p}, \lambda) + \text{CONSTANT} \right\}$$

↳ CONSTANT TERM CAN BE DROPPED  
IF WE DEFINE VACUUM STATE AS  
STATE OF ZERO ENERGY  
(i.e. EXPRESS ENERGY RELATIVE TO VACUUM STATE)

$$\langle 0 | H | 0 \rangle = \sum_{\lambda} \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \cdot \{ \text{CONSTANT} \}$$

BECAUSE  $\langle 0 | a^\dagger(\vec{p}, \lambda) a(\vec{p}, \lambda) | 0 \rangle = 0$

$$\langle 0 | b^\dagger(\vec{p}, \lambda) b(\vec{p}, \lambda) | 0 \rangle = 0$$

↳ NORMAL ORDERING

PRODUCT OF OPERATORS WHERE CREATION OPERATOR STANDS ON LEFT OF ANNIHILATION OPERATOR

$$N(a^\dagger(\bar{p}, s) a(\bar{p}', s')) = a^\dagger(\bar{p}, s) a(\bar{p}', s')$$

$$N(a(\bar{p}, s) a^\dagger(\bar{p}', s')) = -a^\dagger(\bar{p}', s') a(\bar{p}, s)$$

AND ANALOGOUS FOR  $b$

WHEN INTERCHANGING FERMION OPERATORS :  
FOR EVERY INTERCHANGE  $\Rightarrow (-1)$

↳ BY DEFINITION  $\langle 0 | N(A.B) | 0 \rangle$

$\circ_0$   $H = \langle 0 | H | 0 \rangle$

IS EQUIVALENT TO EXPRESSING  $H$  THROUGH NORMAL ORDERED PRODUCT

$$H = \int d^3x \quad N \quad \bar{\Psi}(x) (-i \vec{\gamma} \cdot \vec{\nabla} + m) \Psi(x)$$

↑

IN FOLLOWING : WE WILL EXPRESS ALL QUANTITIES (ENERGY, MOMENTUM, SPIN, CHARGE, ...) RELATIVE TO VACUUM

⇓

NORMAL ORDERING IS UNDERSTOOD IN EXPRESSIONS

↳ ENERGY

$$H = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) + b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

↑  
# PARTICLES  
OF  $(\vec{p}, s)$

↑  
# ANTI-PARTICLE  
OF  $(\vec{p}, s)$

FOR FERMIONS: # PARTICLES = 0, 1

↳ MOMENTUM

ANALOGOUSLY, WE CAN OBTAIN FROM

$$\vec{P} = \int d^3 \vec{x} N \psi^\dagger(x) (-i \vec{\nabla}) \psi(x)$$

↓

$$\vec{P} = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} \vec{p} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) + b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

↳ NOTE: IF WE WOULD HAVE QUANTIZED DIRAC FIELD  
 ACCORDING TO BOSE-EINSTEIN STATISTICS  
 (i.e. THROUGH COMMUTATORS AS FOR KLEIN-GORDON  
 FIELD), WE WOULD HAVE OBTAINED

$$H = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) \ominus b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

DUE TO MINUS SIGN; H WOULD NOT HAVE  
 A LOWER BOUND  $\Rightarrow$  NO STABLE GROUND STATE!  $\nabla$

## ⇒ CHARGE

$$\hookrightarrow \mathcal{L}_{\text{DIRAC}} = \bar{\Psi}(x) (i\gamma^\mu \partial_\mu - m) \Psi(x)$$

CONSIDER GLOBAL PHASE TRANSFORMATION

$$\Psi(x) \rightarrow e^{i\alpha} \Psi(x), \text{ INFINITESIMAL } \delta\Psi = i\alpha\Psi$$

LEAVES  $\mathcal{L}$  INVARIANT :  $\mathcal{L} \rightarrow \mathcal{L}$

∴ SYMMETRY TF.



↳ CONSERVED CURRENT  $\partial_\mu j^\mu = 0$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \Psi)} \cdot \delta\Psi$$

$$j^\mu = \bar{\Psi}(x) \gamma^\mu \Psi(x)$$

↳ CONSERVED CHARGE

$$Q \equiv \int d^3\vec{x} \bar{\Psi}(x) \gamma^0 \Psi(x)$$

$$= \int d^3\vec{x} \Psi^\dagger(x) \Psi(x)$$

↓ BY INSERTING NORMAL MODE EXPANSIONS.

AND CONSIDERING NORMAL ORDERING (EXERCISE!)

$$Q = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) - b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

PARTICLE & ANTI-PARTICLE HAVE OPPOSITE CHARGES!  $\nabla$

### L> ELECTROMAGNETIC CURRENT & CHARGE

BY MULTIPLYING ABOVE  $\mathcal{J}^\mu$  WITH  
ELECTRIC CHARGE  $q$ , WE OBTAIN  
ELECTROMAGNETIC CURRENT  $\mathcal{J}_{EM}^\mu$

$$\mathcal{J}_{EM}^\mu = q \bar{\Psi}(x) \gamma^\mu \Psi(x)$$

CONSERVED CHARGE : ELECTRIC CHARGE

$$Q_{EM} = q \int d^3x \bar{\Psi}^\dagger(x) \Psi(x)$$

e.g. FOR THEORY OF  $e^-$  AND  $e^+$

$$\text{FOR } e^- : q = -e \quad (e > 0, \frac{e^2}{4\pi} \approx \frac{1}{137})$$

$$Q_{EM} = -e \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3} \left\{ a^\dagger(\vec{p}, s) a(\vec{p}, s) - b^\dagger(\vec{p}, s) b(\vec{p}, s) \right\}$$

RELATIVE MINUS SIGN:  $e^-$  AND  $e^+$  HAVE OPPOSITE

ELECTRIC CHARGES:  $q_{e^-} = -e$ ,  $q_{e^+} = +e$

### 3) FEYNMAN PROPAGATOR OF DIRAC FIELD

⇒ 2-POINT CORRELATION FUNCTIONS.

$$\hookrightarrow \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle \equiv i S_{\alpha\beta}^+(x-y)$$

$x, y$  : 2 DIFFERENT SPACE-TIME POINTS

$\alpha, \beta = 1, 2, 3, 4$  : SPINOR INDEX OF DIRAC FIELD

$S_{\alpha\beta}$  :  $4 \times 4$  MATRIX IN DIRAC SPACE

↳ INSERTING NORMAL MODE EXPANSIONS.

$$i S^+(x-y)$$

$$= \sum_{\lambda} \sum_{\lambda'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}}$$

$$\text{AS } b(\vec{p}, \lambda) | 0 \rangle = 0$$

$$\langle 0 | \left( a(\vec{p}, \lambda) u(\vec{p}, \lambda) e^{-ip \cdot x} + \cancel{b(\vec{p}, \lambda)} v(\vec{p}, \lambda) e^{+ip \cdot x} \right)$$

$$\cdot \left( a^\dagger(\vec{p}', \lambda') \bar{u}(\vec{p}', \lambda') e^{+ip' \cdot y} + \cancel{b(\vec{p}', \lambda')} \bar{v}(\vec{p}', \lambda') e^{-ip' \cdot y} \right) | 0 \rangle$$

$$= \sum_{\lambda} \sum_{\lambda'} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} e^{-ip \cdot x} e^{ip' \cdot y} u(\vec{p}, \lambda) \bar{u}(\vec{p}', \lambda')$$

$$\cdot \langle 0 | a(\vec{p}, \lambda) a^\dagger(\vec{p}', \lambda') | 0 \rangle$$

$$\begin{aligned}
 & \langle 0 | a(\bar{p}, s) a^\dagger(\bar{p}', s') | 0 \rangle \\
 &= \langle 0 | [a(\bar{p}, s), a^\dagger(\bar{p}', s')]_+ | 0 \rangle \\
 &= (2\pi)^3 \delta^3(\bar{p} - \bar{p}') \delta_{ss'}
 \end{aligned}$$

$$\begin{aligned}
 iS^+(x-y) &= \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{-ip \cdot (x-y)} \Big|_{p^0 = E_{\bar{p}}} \\
 &\quad \cdot \underbrace{\sum_s U(\bar{p}, s) \bar{U}(\bar{p}, s)}_{(\not{p} + m)} \quad \text{SPIN SUM}
 \end{aligned}$$

$$\therefore iS^+(x-y) = \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{-ip \cdot (x-y)} (\not{p} + m) \Big|_{p^0 = E_{\bar{p}}}$$

$$\hookrightarrow iS^+(x) = (i\gamma^\mu \partial_\mu + m) \underbrace{\int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{-ip \cdot x}}_{\text{SCALAR CORRELATOR FROM KLEIN-GORDON THEORY } i\Delta(x)} \Big|_{p^0 = E_{\bar{p}}}$$

SCALAR CORRELATOR  
FROM KLEIN-GORDON  
THEORY  $i\Delta(x)$

$$S^+(x) = (i\gamma^\mu \partial_\mu + m) \Delta(x)$$

↳ ANALOGOUSLY :

$$\underline{\underline{\langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle \equiv i S_{\beta\alpha}^-(x-y)}}$$

$$i S_{\beta\alpha}^-(x-y)$$

$$= \sum_s \sum_{s'} \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} e^{i p \cdot x} e^{-i p' \cdot y}$$

$$\cdot \bar{v}_\beta(\bar{p}', s') v_\alpha(\bar{p}, s)$$

$$\cdot \langle 0 | b(\bar{p}', s') b^\dagger(\bar{p}, s) | 0 \rangle$$

$$= \int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{i p \cdot (x-y)} \underbrace{\sum_s v_\alpha(\bar{p}, s) \bar{v}_\beta(\bar{p}, s)}_{(\not{p} - m)_{\alpha\beta}} \Big|_{p^0 = E_{\bar{p}}}$$

$$\boxed{i S_{\beta\alpha}^-(x-y) = \int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{i p \cdot (x-y)} (\not{p} - m)_{\alpha\beta} \Big|_{p^0 = E_{\bar{p}}}}$$

$$i S_{\beta\alpha}^-(x) = - (i \gamma^\mu \partial_\mu + m)_{\alpha\beta} \underbrace{\int \frac{d^3 \bar{p}}{(2\pi)^3 2E_{\bar{p}}} e^{i p \cdot x}}_{i \Delta(-x)}$$

$$\boxed{S_{\beta\alpha}^-(x) = - (i \gamma^\mu \partial_\mu + m)_{\alpha\beta} \Delta(-x)}$$



## ⇒ TIME-ORDERED PRODUCT FOR FERMION FIELDS

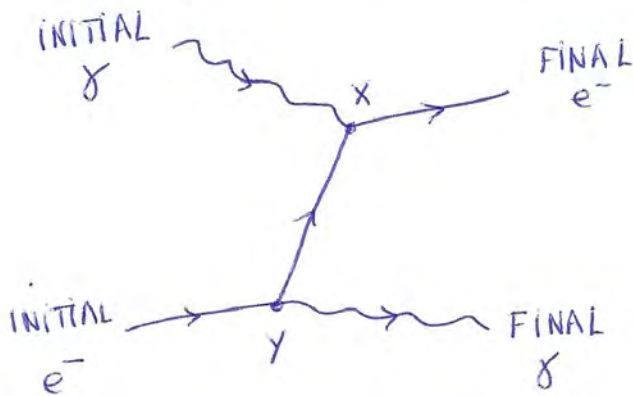
$$T \psi(x) \bar{\psi}(y) \equiv \psi(x) \bar{\psi}(y) \theta(x^0 - y^0) - \bar{\psi}(y) \psi(x) \theta(y^0 - x^0)$$

NOTE! - SIGN BECAUSE OF ANTI-COMMUTATION OF FERMION FIELDS

## ⇒ FEYNMAN PROPAGATOR FOR SPIN 1/2

↳ ENTERS PHYSICAL AMPLITUDES

e.g. COMPTON SCATTERING ON  $e^-$  : 2 CONTRIBUTIONS

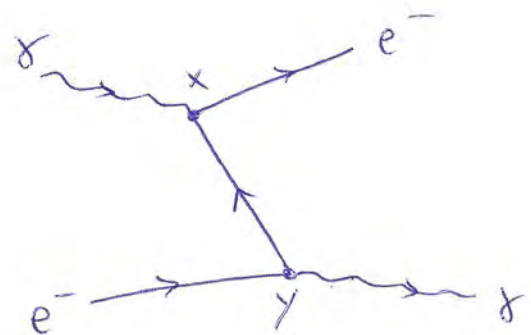


→ TIME

$$x^0 > y^0$$

PROPAGATION OF PARTICLE FROM  $y \rightarrow x$

$e^-$  PROPAGATES FORWARD IN TIME FROM  $y \rightarrow x$



→ TIME

$$y^0 > x^0$$

PROPAGATION OF ANTI-PARTICLE  $x \rightarrow y$

- AT  $x$  :  $e^- e^+$  PAIR IS CREATED
- $e^+$  PROPAGATES FORWARD IN TIME FROM  $x \rightarrow y$
- AT  $y$  :  $e^+$  ANNIHILATES WITH INITIAL  $e^-$

↳ DEFINITION :

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \equiv i S_F(x-y)$$

$$\begin{aligned} (i S_F(x-y))_{\alpha\beta} &= \theta(x^0 - y^0) \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle \\ &\quad - \theta(y^0 - x^0) \langle 0 | \bar{\psi}_\beta(y) \psi_\alpha(x) | 0 \rangle \\ &= \theta(x^0 - y^0) i S_{\alpha\beta}^+(x-y) \\ &\quad - \theta(y^0 - x^0) i S_{\beta\alpha}^-(x-y) \end{aligned}$$

$$\begin{aligned} (S_F(x))_{\alpha\beta} &= \theta(x^0) \cdot (i \gamma^\mu \partial_\mu + m)_{\alpha\beta} \Delta(x) \\ &\quad - \theta(-x^0) \cdot (i \gamma^\mu \partial_\mu + m)_{\beta\alpha} (-1) \Delta(-x) \end{aligned}$$

$$S_F(x) = (i \gamma^\mu \partial_\mu + m) \underbrace{(\theta(x^0) \Delta(x) + \theta(-x^0) \Delta(-x))}_{\Delta_F(x)}$$

$$S_F(x) = (i \gamma^\mu \partial_\mu + m) \Delta_F(x)$$

WITH  $\Delta_F$  : SPIN-0 (KLEIN-GORDON) FEYNMAN PROPAGATOR

↳ FEYNMAN PROPAGATOR IN MOMENTUM SPACE

$$S_F(x) \equiv \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} \tilde{S}_F(p)$$

WHERE  $\tilde{S}_F(p)$  IS FEYNMAN PROPAGATOR FOR SPIN 1/2 PARTICLE IN MOMENTUM SPACE

(NOTE: IN FOLLOWING WE WILL FOR SIMPLICITY DROP  $\sim$  ON  $\tilde{S}_F(p)$  WHEN IT IS UNDERSTOOD THAT WE WORK IN MOMENTUM SPACE)

↳ EXPLICIT FORM

$$\begin{aligned}
 S_F(x) &= (i \gamma^\mu \partial_\mu + m) \Delta_F(x) \\
 &= (i \gamma^\mu \partial_\mu + m) \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} \frac{1}{p^2 - m^2 + i\epsilon} \\
 &= \int \frac{d^4 p}{(2\pi)^4} e^{-i p \cdot x} \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}
 \end{aligned}$$

∴

$$\tilde{S}_F(p) = \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

# 4) SYMMETRIES OF THE DIRAC THEORY

## ⇒ LORENTZ INVARIANCE

↳ EQUATIONS OF MOTION SHOULD BE SAME IN ANY INERTIAL FRAME

LORENTZ TF:  $x^\mu \rightarrow x'^\mu = a^\mu_\nu x^\nu$

WITH  $\underline{a^\mu_\nu a^\lambda_\mu = g^\lambda_\nu}$

↓  
ENSURES THAT  $x'^\mu x'_\mu = x^\mu x_\mu$   
SPACE-TIME INTERVAL IS SAME FOR ALL INERTIAL OBSERVERS

$$\det(a^\mu_\nu) = \pm 1$$

↗  $\det(a^\mu_\nu) = +1$

PROPER LORENTZ TF  
(3 ROTATIONS,  
3 BOOSTS)

↘  $\det(a^\mu_\nu) = -1$

IMPROPER LORENTZ TF

↘  
COMBINATIONS OF PROPER LORENTZ TF WITH

SPACE INVERSION

OR

TIME REVERSAL

$$a^\mu_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$a^\mu_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

↳ TF. FOR SPINOR FIELD

$$x^\mu \xrightarrow{a} x'^\mu = a^\mu_\nu x^\nu$$

$$\underline{\underline{\psi'(x') = S(a) \psi(x)}}$$

LINEAR TF.

IN MATRIX NOTATION  $x' = a x$   
 $x = a^{-1} x'$

$$\underline{\underline{\psi(x) \xrightarrow{a} \psi'(x) = S(a) \psi(a^{-1}x)}}$$

INVERSE TF  $\psi(x) = S^{-1}(a) \psi'(ax)$   
 $= S(a^{-1}) \psi'(ax)$

$$\underline{\underline{S^{-1}(a) = S(a^{-1})}}$$

DIRAC EQ. IS SAME FOR BOTH INERTIAL OBSERVERS  
 (COVARIANCE)

$$\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) \psi(x) = 0$$

↓

$$\left( i \gamma^\mu \frac{\partial}{\partial x^\mu} - m \right) S^{-1}(a) \psi'(x') = 0$$

↓ MULTIPLY ON LEFT  
 BY  $S(a)$

$$\left( i S(a) \gamma^\mu S^{-1}(a) \frac{\partial}{\partial x^\mu} - m \right) \psi'(x') = 0$$

↓

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = a^\nu_\mu \frac{\partial}{\partial x'^\nu}$$

$$\left( i S(a) \gamma^\mu S^{-1}(a) a^\nu{}_\mu \frac{\partial}{\partial x'^\nu} - m \right) \psi'(x') = 0$$

LORENTZ COVARIANCE REQUIRES

$$\left( i \gamma^\nu \frac{\partial}{\partial x'^\nu} - m \right) \psi'(x') = 0$$

$$\Downarrow$$

$$S(a) \gamma^\mu S^{-1}(a) a^\nu{}_\mu = \gamma^\nu$$

$$\Downarrow$$

$$S^{-1}(a) \gamma^\nu S(a) = a^\nu{}_\mu \gamma^\mu$$

↳ EXPLICIT CONSTRUCTION OF  $S(a)$  FOR PROPER LORENTZ TF.

- CONSIDER INFINITESIMAL PROPER LORENTZ TF.

$$a^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$$

$$(a^{-1})^\mu{}_\nu = g^\mu{}_\nu - \omega^\mu{}_\nu$$

FOR LORENTZ TF  $a^\mu{}_\nu g_\mu{}^\lambda = g_\nu{}^\lambda$

$$\left( g^\mu{}_\nu + \omega^\mu{}_\nu \right) \left( g_\mu{}^\lambda + \omega_\mu{}^\lambda \right) = g_\nu{}^\lambda$$

$$\omega^\lambda{}_\nu + \omega_\nu{}^\lambda = 0 \Rightarrow \underline{\underline{\omega_{\mu\nu} = -\omega_{\nu\mu}}}$$

ANTI-SYMMETRIC

- FOR LINEAR TF :  $S(a)$  CAN BE CONSTRUCTED AS LINEAR IN  $\omega^{\mu\nu}$

$$S(a) = \mathbb{1} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}$$

WITH  $\sigma_{\mu\nu}$  :  $4 \times 4$  ANTI-SYMM MATRICES

$$\sigma_{\mu\nu} = -\sigma_{\nu\mu}$$

6 INDEPENDENT :  $\begin{cases} \nearrow 3 \text{ ROTATIONS} \\ \searrow 3 \text{ BOOSTS.} \end{cases}$

- $\sigma_{\mu\nu}$  DETERMINED FROM

$$S^{-1}(a) \gamma^\mu S(a) = a^\mu{}_\nu \gamma^\nu$$

$$\left[ \mathbb{1} + \frac{i}{4} \sigma_{\alpha\beta} \omega^{\alpha\beta} \right] \gamma^\mu \left[ \mathbb{1} - \frac{i}{4} \sigma_{\alpha\beta} \omega^{\alpha\beta} \right] = \gamma^\mu + \omega^\mu{}_\nu \gamma^\nu$$

$\Downarrow$  KEEPING TERMS LINEAR IN  $\omega$

$$\frac{i}{4} \left[ \sigma_{\alpha\beta}, \gamma^\mu \right] \omega^{\alpha\beta} = \omega^{\mu\beta} \gamma_\beta = \frac{1}{2} \omega^{\alpha\beta} \left( g^\mu{}_\alpha \gamma_\beta - g^\mu{}_\beta \gamma_\alpha \right)$$

$\uparrow$  BECAUSE OF ANTI-SYMM. OF  $\omega^{\alpha\beta}$

IN ORDER FOR THIS TO HOLD  $\forall \omega^{\alpha\beta}$  :

$$\underline{\underline{[\sigma_{\alpha\beta}, \gamma^\mu] = -2i (g_\alpha^\mu \gamma_\beta - g_\beta^\mu \gamma_\alpha)}}$$

- THIS EQUATION IS SOLVED BY

$$\sigma_{\alpha\beta} = \frac{i}{2} [\gamma_\alpha, \gamma_\beta] = \frac{i}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)$$

PROOF :

$$\begin{aligned} & [\sigma_{\alpha\beta}, \gamma^\mu] \\ &= \sigma_{\alpha\beta} \gamma^\mu - \gamma^\mu \sigma_{\alpha\beta} \\ &= \frac{i}{2} \left\{ (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \gamma^\mu - \gamma^\mu (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \right\} \\ &= \frac{i}{2} \left\{ (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha) \gamma^\mu - (2g_\alpha^\mu - \gamma_\alpha \gamma^\mu) \gamma_\beta \right. \\ &\quad \left. + (2g_\beta^\mu - \gamma_\beta \gamma^\mu) \gamma_\alpha \right\} \\ &= \frac{i}{2} \left\{ \cancel{(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)} \gamma^\mu - 2g_\alpha^\mu \gamma_\beta + \gamma_\alpha (2g_\beta^\mu - \cancel{\gamma_\beta \gamma^\mu}) \right. \\ &\quad \left. + 2g_\beta^\mu \gamma_\alpha - \gamma_\beta (2g_\alpha^\mu - \cancel{\gamma_\alpha \gamma^\mu}) \right\} \end{aligned}$$

$$\nabla \circ = 2i (g_\beta^\mu \gamma_\alpha - g_\alpha^\mu \gamma_\beta)$$



◦◦ UNDER PROPER LORENTZ TF

$$x \xrightarrow{a} x' = a x$$

$$\Psi(x) \rightarrow \Psi'(x') = S(a) \Psi(a^{-1}x)$$

↳ INFINITESIMAL  $a^u_v = g^u_v + \omega^u_v$

$$S(a) = \mathbb{1} - \frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}$$

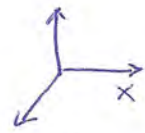
WITH  $\sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$

↳ FINITE PROPER LORENTZ TF

$$S(a) = \exp \left\{ -\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu} \right\}$$

WITH  $\omega^{\mu\nu}$  FINITE

↳ LORENTZ BOOST ALONG X-AXIS



$$\bullet a^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\left. \begin{aligned} \gamma &= \cosh \omega \\ \beta\gamma &= \sinh \omega \end{aligned} \right\} \beta = \tanh \omega$$

• INFINITESIMAL TF

$$\gamma \approx 1, \quad \Delta\beta \ll 1$$

$$\omega^{\mu}_{\nu} \approx (\Delta\beta) \begin{pmatrix} 0 & -1 & & \\ -1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \equiv (\Delta\beta) (\underline{I}_1)^{\mu}_{\nu}$$

$$\underline{I}_1^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$\underline{I}_1^{2m+1} = \underline{I}_1 \quad m = 1, 2, 3, \dots$$

$$\underline{I}_1^{2m} = \underline{I}_1^2 \quad m = 2, 3, 4, \dots$$

$$(\underline{I}_1)^{0,1} = 1$$

$$(\underline{I}_1)^{1,0} = -1$$

- FINITE LORENTZ BOOST CAN BE OBTAINED FROM SUCCESSION OF INFINITESIMAL BOOSTS  $\Delta\beta = \frac{\omega}{N}$

$$a^{\mu}_{\nu} = \left( \lim_{N \rightarrow \infty} \left( \mathbb{1} + \frac{\omega}{N} \mathbb{I}_1 \right)^N \right)^{\mu}_{\nu}$$

$$= \left( \exp \left\{ \omega \mathbb{I}_1 \right\} \right)^{\mu}_{\nu}$$

$$\exp \left\{ \omega \mathbb{I}_1 \right\} = \left[ \mathbb{1} + \frac{\omega^2}{2!} \mathbb{I}_1^2 + \frac{\omega^4}{4!} \mathbb{I}_1^4 + \dots \right]$$

$$+ \left[ \omega \mathbb{I}_1 + \frac{\omega^3}{3!} \mathbb{I}_1^3 + \dots \right]$$

$$= \mathbb{1} - \mathbb{I}_1^2 + (\cosh \omega) \mathbb{I}_1^2 + (\sinh \omega) \mathbb{I}_1$$

$$= \begin{pmatrix} \cosh \omega & -\sinh \omega & & \\ -\sinh \omega & \cosh \omega & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- INFINITESIMAL BOOST (ALONG x-AXIS)

$$S(a) = \mathbb{1} - \frac{i}{2} \sigma_{01} (\Delta\beta) (\mathbb{I}_1)^{01}$$

$$\sigma_{01} = -i \begin{pmatrix} 0 & \alpha^1 \\ \sigma^1 & 0 \end{pmatrix} = -i \alpha^1$$

$$\underline{\underline{S(a) = \mathbb{1} - \frac{1}{2} (\Delta\beta) \alpha^1}}$$

- FINITE BOOST (ALONG x-AXIS)

$$\Delta\beta = \frac{\omega}{N}$$

$$S(a) = \lim_{N \rightarrow \infty} \left( \mathbb{1} - \frac{i}{2} \sigma_{01} \frac{\omega}{N} (\mathbb{I}_1)^{01} \right)^N$$

$$= \exp \left\{ - \frac{i}{2} \sigma_{01} \underbrace{\omega (\mathbb{I}_1)^{01}}_{\omega \alpha^1} \right\}$$

$$\underline{\underline{S(a) = \exp \left\{ - \frac{1}{2} \omega \alpha^1 \right\}}}$$

↳ ROTATION AROUND Z-AXIS OVER ANGLE  $\varphi$

$$\bullet \quad a^u_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi & 0 \\ 0 & \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

• INFINITESIMAL TF

$$\Delta\varphi \ll \cos \Delta\varphi \approx 1$$

$$\sin \Delta\varphi \approx \Delta\varphi$$

$$\omega^u_v = (\Delta\varphi) i \begin{pmatrix} 0 & & & \\ & 0 & +i & \\ & -i & 0 & \\ & & & 0 \end{pmatrix} \equiv i(\Delta\varphi) (R_3)^u_v$$

$$R_3^{2n} = R_3^2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

$$R_3^{2n+1} = R_3$$

$$(R_3)^{12} = -i$$

$$(R_3)^{21} = i$$

- FINITE ROTATION OBTAINED AS SUCCESSION OF INFINITESIMAL ROTATIONS  $\Delta\varphi = \frac{\varphi}{N}$

$$a = \lim_{N \rightarrow \infty} \left( \mathbb{I} + \frac{i\varphi}{N} R_3 \right)^N$$

$$= \exp \{ i\varphi R_3 \}$$

$$= \mathbb{I} - R_3^2 + \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{2n}}{(2n)!} R_3^{2n}$$

$$+ i \sum_{n=0}^{\infty} (-1)^n \frac{\varphi^{2n+1}}{(2n+1)!} R_3^{2n+1}$$

$$= \mathbb{I} + (\cos \varphi - 1) R_3^2 + i \sin \varphi R_3$$

$$= \begin{pmatrix} 1 & & & \\ & \cos \varphi & -\sin \varphi & \\ & \sin \varphi & \cos \varphi & \\ & & & 1 \end{pmatrix}$$

- INFINITESIMAL ROTATION (AROUND Z-AXIS)

$$S(a) = \mathbb{1} - \frac{i}{2} \sigma_{12} (i \Delta\varphi) (\mathbb{R}_3)^{12}$$

$$\begin{aligned} \sigma_{12} &= \frac{i}{2} (\gamma^1 \gamma^2 - \gamma^2 \gamma^1) = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \\ &\equiv \Sigma^3 \end{aligned}$$

$$\underline{\underline{S(a) = \mathbb{1} - \frac{i}{2} (\Delta\varphi) \Sigma^3}}$$

- FINITE ROTATION (AROUND Z-AXIS)  $\Delta\varphi = \frac{\varphi}{N}$

$$S(a) = \lim_{N \rightarrow \infty} \left( \mathbb{1} - \frac{i}{2} \sigma_{12} i \frac{\varphi}{N} (\mathbb{R}_3)^{12} \right)^N$$

$$= \exp \left\{ - \frac{i}{2} \sigma_{12} \underbrace{\varphi (i \mathbb{R}_3)^{12}}_{\equiv (\omega)^{12}} \right\}$$

$$\underline{\underline{S(a) = \exp \left\{ - \frac{i}{2} \varphi \Sigma^3 \right\}}}$$

- CONSIDER SPINOR WITH SPIN PROJECTION  $+\frac{1}{2}$  ALONG Z-AXIS

$$i.e. \sum^3 \psi = + \psi$$



UNDER ROTATION, THIS SPINOR TRANSFORMS AS

$$\psi'(x') = \exp\left(-\frac{i}{2}\varphi\right) \psi(x)$$

IT TAKES A ROTATION OVER  $\underline{4\pi}$  (!) BEFORE SPINOR TURNS INTO ITSELF



PHYSICAL QUANTITIES MUST BE BILINEARS IN  $\psi$   
 ↳ TURN INTO THEMSELVES AFTER ROTATION OVER  $2\pi$

- NOTE: → FOR ROTATIONS.

$$\sigma_{ij}^{\dagger} = \sigma_{ij} \quad \Rightarrow \quad \underset{\substack{\uparrow \\ \text{ROTATION}}}{S_R^{\dagger}} = S_R^{-1}$$

→ FOR BOOSTS

$$S_B = \exp\left\{-\frac{1}{2}\omega \alpha^1\right\}$$

$$S_B^{\dagger} = S_B$$

• FOR PROPER L.T:  $S^{-1} = \gamma_0 S^{\dagger} \gamma_0$



↳ TRANSFORMATION PROPERTIES UNDER LORENTZ TF  
OF CURRENT

$$J^\mu(x) = \bar{\Psi}(x) \gamma^\mu \Psi(x)$$

↓

$$J'^\mu(x') = \bar{\Psi}'(x') \gamma^\mu \Psi'(x')$$

$$= \bar{\Psi}(x) \underbrace{S^\dagger \gamma^0 \gamma^\mu S}_{\gamma^\mu} \Psi(x)$$

$$= \bar{\Psi}(x) \gamma^0 \underbrace{S^{-1} \gamma^\mu S}_{a^\mu{}_\nu \gamma^\nu} \Psi(x)$$

$$= a^\mu{}_\nu \bar{\Psi}(x) \gamma^\nu \Psi(x)$$

$$\underline{J'^\mu(x') = a^\mu{}_\nu J^\nu(x)}$$

$J^\mu$  TRANSFORMS AS A 4-VECTOR

↳ TRANSFORMATION OF FIELD OPERATORS  
IN HILBERT SPACE UNDER PROPER L.T. a

- STATE  $|\bar{p}, s\rangle = \sqrt{2E_{\bar{p}}} a^\dagger(\bar{p}, s) |0\rangle$   
 CONSIDER SPIN QUANTIZATION AXIS EITHER ALONG  $\left\{ \begin{array}{l} \text{BOOST AXIS} \\ \text{ROTATION AXIS} \end{array} \right.$   
 $|\bar{p}, s\rangle \xrightarrow{a} |a\bar{p}, s\rangle = \sqrt{2E_{a\bar{p}}} a^\dagger(a\bar{p}, s) |0\rangle$

NORM HAS TO BE INVARIANT  $\Rightarrow$  UNITARY T.F. U  
 $(U^\dagger = U^{-1})$

$$|a\bar{p}, s\rangle = U(a) |\bar{p}, s\rangle$$

$$\Downarrow$$

$$\sqrt{2E_{a\bar{p}}} a^\dagger(a\bar{p}, s) = U(a) a^\dagger(\bar{p}, s) U^\dagger(a) \sqrt{2E_{\bar{p}}}$$

$$\left\| \begin{array}{l} a(\bar{p}, s) \xrightarrow{a} U(a) a(\bar{p}, s) U^\dagger(a) \\ = \sqrt{\frac{E_{a\bar{p}}}{E_{\bar{p}}}} a(a\bar{p}, s) \end{array} \right.$$

• FIELD OPERATOR

$$\Psi(x) \xrightarrow{a} U(a) \Psi(x) U^\dagger(a)$$

$$= \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ U(a) a(\bar{p}, s) U^\dagger(a) U(\bar{p}, s) e^{-ip \cdot x} \right. \\ \left. + U(a) b^\dagger(\bar{p}, s) U^\dagger(a) \psi(\bar{p}, s) e^{+ip \cdot x} \right\}$$

$$= \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3 2E_{\bar{p}}} \sqrt{2E_{a\bar{p}}} \left\{ a(a\bar{p}, s) U(\bar{p}, s) e^{-ip \cdot x} \right. \\ \left. + b^\dagger(a\bar{p}, s) \psi(\bar{p}, s) e^{+ip \cdot x} \right\}$$

CHANGE OF VARIABLE  $\bar{p}' = a \bar{p}$ 

$$\frac{d^3 \bar{p}'}{2E_{\bar{p}'}} = \frac{d^3 \bar{p}}{2E_{\bar{p}}}$$

$$p \cdot x = p' \cdot x' \quad \text{WITH } x' = ax$$

$$U(a) \psi(x) U^\dagger(a)$$

$$= \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}', s) U(a^{-1} \bar{p}', s) e^{-ip' \cdot x'} + b^\dagger(\bar{p}', s) v(a^{-1} \bar{p}', s) e^{+ip' \cdot x'} \right\}$$

$$\downarrow \quad U(a^{-1} \bar{p}', s) = S^{-1}(a) U(\bar{p}', s)$$

(SEE PROOF P. 52-53)

$$= \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}', s) S^{-1}(a) U(\bar{p}', s) e^{-ip' \cdot x'} + b^\dagger(\bar{p}', s) S^{-1}(a) v(\bar{p}', s) e^{+ip' \cdot x'} \right\}$$

$$= S^{-1}(a) \psi(x')$$

$$\therefore \underline{\underline{U(a) \psi(x) U^\dagger(a) = S^{-1}(a) \psi(ax)}}$$

NOTE: THIS IS RELATION FOR AN ACTIVE LORENTZ TF  $a$ .

CHECK

$$\underline{\underline{U(a\bar{p}, 1) = S(a) U(\bar{p}, 1)}}$$

FOR LORENTZ BOOST ALONG X-AXIS  
WITH SPIN QUANTIZATION AXIS CHOSEN ALONG X-AXIS.  
(WE CONSIDER HERE AN ACTIVE TF.)

↳ INVERSE OF PASSIVE TF.  
(COORDINATE TF.)

$$a = \begin{pmatrix} \cosh w & \sinh w & & \\ \sinh w & \cosh w & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

FOR PARTICLE AT REST, WHICH IS BOOSTED ALONG X-AXIS

$$\begin{pmatrix} E_{\bar{p}} \\ P_x \end{pmatrix} = \begin{pmatrix} \cosh w & \sinh w \\ \sinh w & \cosh w \end{pmatrix} \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$\left. \begin{aligned} \cosh w &= \frac{E_{\bar{p}}}{m} \\ \sinh w &= \frac{P_x}{m} = \frac{|\bar{p}|}{m} \end{aligned} \right\} \tanh w = \frac{|\bar{p}|}{E_{\bar{p}}}$$

$$\downarrow$$

$$\cosh \frac{w}{2} = \sqrt{\frac{1 + \cosh w}{2}} = \sqrt{\frac{E_{\bar{p}} + m}{2m}}$$

$$\begin{aligned} \sinh \frac{w}{2} &= \frac{\cosh w}{2 \cosh \frac{w}{2}} \tanh w = \frac{|\bar{p}|}{2m} \cdot \sqrt{\frac{2m}{E_{\bar{p}} + m}} \\ &= \frac{|\bar{p}|}{E_{\bar{p}} + m} \sqrt{\frac{E_{\bar{p}} + m}{2m}} \end{aligned}$$

PROOF  $U(\bar{P}, \gamma) = S(a) U(\bar{0}, \gamma)$

$\hookrightarrow S(a) = \exp \left\{ \frac{1}{2} \omega \alpha^1 \right\}$  (ACTIVE TF)

$$= \sum_{n=0}^{\infty} \left( \frac{\omega}{2} \right)^{2n} (\alpha^1)^{2n} + \sum_{n=0}^{\infty} \left( \frac{\omega}{2} \right)^{2n+1} (\alpha^1)^{2n+1}$$

$$(\alpha_1)^{2n} = \alpha_1^2 = \mathbb{I}, \quad n=0, 1, \dots$$

$$(\alpha_1)^{2n+1} = \alpha_1, \quad n=0, 1, \dots$$

$$= \cosh \frac{\omega}{2} \mathbb{I} + \sinh \frac{\omega}{2} \alpha^1$$

$$= \left( \begin{array}{c|c} \cosh \frac{\omega}{2} \mathbb{I}_{2 \times 2} & \sinh \frac{\omega}{2} \sigma^1 \\ \hline \sinh \frac{\omega}{2} \sigma^1 & \cosh \frac{\omega}{2} \mathbb{I}_{2 \times 2} \end{array} \right)$$

$\hookrightarrow S(a) U(\bar{0}, \gamma)$

$$= \left( \begin{array}{c|c} \cosh \frac{\omega}{2} \mathbb{I} & \sinh \frac{\omega}{2} \sigma^1 \\ \hline \sinh \frac{\omega}{2} \sigma^1 & \cosh \frac{\omega}{2} \mathbb{I} \end{array} \right) \begin{pmatrix} X_s \\ 0 \end{pmatrix} \sqrt{2m}$$

$$= \sqrt{2m} \cosh \frac{\omega}{2} \begin{pmatrix} X_s \\ \tanh \frac{\omega}{2} \sigma^1 X_s \end{pmatrix}$$

$$= \sqrt{E_{\bar{p}} + m} \begin{pmatrix} X_s \\ \frac{\sigma^1 |\bar{p}|}{E_{\bar{p}} + m} X_s \end{pmatrix} \stackrel{!}{=} U(\bar{P}, \gamma)$$

$\hookrightarrow \sigma^1 |\bar{p}| = \bar{v} \cdot \bar{p}$  (BOOST ALONG x)

## ⇒ PARITY (SPATIAL INVERSION)

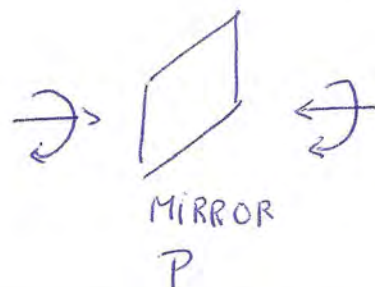
↳ SPATIAL INVERSION ( $P$ ): IMPROPER LORENTZ TF WHICH INVERTS SPATIAL COORDINATES

$$(t, \bar{x}) \xrightarrow{P} (t, -\bar{x})$$

$$\det a = -1$$

• 3-MOMENTUM CHANGES SIGN

$$\bar{p} \xrightarrow{P} -\bar{p}$$



• ANGULAR MOMENTUM / SPIN REMAINS UNCHANGED

$$\bar{L} = \bar{x} \times \bar{p} \xrightarrow{P} (-\bar{x}) \times (-\bar{p}) = \bar{x} \times \bar{p} = \bar{L}$$

↳ APPLICATION OF P ON SPINOR

$$\psi'(x') = P \psi(x)$$

IF WE APPLY TWICE: STATE GOES BACK TO ITSELF

$$P^2 = 1 \quad \rightarrow \quad P^{-1} = P$$

$$P \gamma^\nu P = (a_P)^\nu_\mu \gamma^\mu \quad \text{WITH } a_P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$P \gamma^0 P = \gamma^0$$

$$P \gamma^i P = -\gamma^i \quad i = 1, 2, 3$$

⇓

$$\underline{P = \eta \gamma^0}$$

$$\eta = \pm 1 \quad \text{PHASE FACTOR}$$

L> TRANSFORMATION OF FIELD OPERATOR

WE WILL PROVE:  $\underline{\Psi(t, \bar{x}) \xrightarrow{P} \eta \gamma^0 \Psi(t, -\bar{x}) \equiv U(a_p) \Psi(x) U(a_p)}$

STARTING FROM:

$$a(\bar{p}, s) \xrightarrow{P} \eta_a a(-\bar{p}, s)$$

$$b(\bar{p}, s) \xrightarrow{P} \eta_b b(-\bar{p}, s)$$

$$U^{-1}(a_p) = U^\dagger(a_p) = U(a_p)$$

$$\Psi(x) = \sum_s \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ a(\bar{p}, s) u(\bar{p}, s) e^{-i p \cdot x} + b^\dagger(\bar{p}, s) v(\bar{p}, s) e^{+i p \cdot x} \right\}$$

$$\Psi(t, \bar{x}) \xrightarrow{P} \sum_s \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ \eta_a a(-\bar{p}, s) u(\bar{p}, s) e^{-i p \cdot x} + \eta_b^* b^\dagger(-\bar{p}, s) v(\bar{p}, s) e^{+i p \cdot x} \right\}$$

$$\downarrow \quad \bar{p} \rightarrow \bar{p}' = -\bar{p}$$

$$= \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ \eta_a a(\bar{p}', s) u(-\bar{p}', s) e^{-i E_{\bar{p}'} t - i \bar{p}' \cdot \bar{x}} + \eta_b^* b^\dagger(\bar{p}', s) v(-\bar{p}', s) e^{i E_{\bar{p}'} t + i \bar{p}' \cdot \bar{x}} \right\}$$

$$U(-\vec{p}', s) = N \begin{pmatrix} \chi_s \\ -\frac{\vec{\sigma} \cdot \vec{p}'}{E_{\vec{p}'} + m} \chi_s \end{pmatrix} = \gamma^0 U(\vec{p}', s)$$

$$v(-\vec{p}', s) = N \begin{pmatrix} -\frac{\vec{\sigma} \cdot \vec{p}'}{E_{\vec{p}'} + m} \chi'_s \\ \chi'_s \end{pmatrix} = -\gamma^0 v(\vec{p}', s)$$

$$\Psi(t, \vec{x}) \xrightarrow{P} \sum_s \int \frac{d^3 \vec{p}'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}} \left\{ \eta_a a(\vec{p}', s) \gamma^0 U(\vec{p}', s) e^{-ip' \cdot x'} \right. \\ \left. - \eta_b^* b^\dagger(\vec{p}', s) \gamma^0 v(\vec{p}', s) e^{+ip' \cdot x'} \right\}$$

$$\text{WITH } x' = (t, -\vec{x})$$

$$\text{CHOOSE } \underline{\underline{\eta_b^* = -\eta_a}}$$

(PARTICLE & ANTI-PARTICLE HAVE OPPOSITE INTRINSIC PARITIES)

$$\Psi(t, \vec{x}) \xrightarrow{P} \eta_a \gamma^0 \Psi(t, -\vec{x})$$

$$\bar{\Psi}(t, \vec{x}) \xrightarrow{P} \bar{\Psi}(t, -\vec{x}) \gamma^0 \eta_a^*$$

$$\text{WITH } |\eta_a|^2 = 1 \quad \eta_a = \pm 1$$

WE CAN CONVENIENTLY CHOOSE  $\eta_a = +1$



## ↳ TRANSFORMATION OF BILINEARS

- SCALAR

$$\bar{\Psi}(x) \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma^0 \Psi(x')$$

$$= \bar{\Psi}(x') \Psi(x')$$

WITH  $x' = (t, -\bar{x})$

- PSEUDO-SCALAR

$$\bar{\Psi}(x) \gamma_5 \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma_5 \gamma^0 \Psi(x')$$

$$= - \bar{\Psi}(x') \gamma_5 \Psi(x')$$

↑  
CHANGES SIGN UNDER P

- VECTOR

$$\bar{\Psi}(x) \gamma^\mu \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma^\mu \gamma^0 \Psi(x')$$

$$= \begin{cases} + \bar{\Psi}(x') \gamma^0 \Psi(x') & , \mu = 0 \\ - \bar{\Psi}(x') \gamma^i \Psi(x') & , \mu = i = 1, 2, 3 \end{cases}$$

TRANSFORMS AS 4-VECTOR UNDER P

↳ TIME COMPONENT  
DOES NOT CHANGE SIGN

↳ SPACE COMPONENTS  
CHANGE SIGN UNDER P

- AXIAL - VECTOR

$$\bar{\Psi}(x) \gamma^{\mu} \gamma_5 \Psi(x) \xrightarrow{P} |M_a|^2 \bar{\Psi}(x') \gamma^0 \gamma^{\mu} \gamma_5 \gamma^0 \Psi(x')$$

$$= \begin{cases} - \bar{\Psi}(x') \gamma^0 \gamma_5 \Psi(x') , & \mu = 0 \\ + \bar{\Psi}(x') \gamma^i \gamma_5 \Psi(x') , & \mu = i = 1, 2, 3 \end{cases}$$

↳ TIME COMPONENT CHANGES SIGN

↳ SPACE COMPONENTS DO NOT CHANGE SIGN UNDER P

(e.g. ANGULAR MOMENTUM  $\vec{L}$ )

- TENSOR

$$\bar{\Psi}(x) \sigma^{\mu\nu} \Psi(x) \xrightarrow{P} \bar{\Psi}(x') \gamma^0 \sigma^{\mu\nu} \gamma^0 \Psi(x')$$

$$= \begin{cases} 0 & \mu = 0, \nu = 0 \\ - \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu = 0, \nu = i \\ - \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu = i, \nu = 0 \\ \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu = i, \nu = j \\ & i \neq j \end{cases}$$

$$= (-1)^{\delta_{\mu i}} (-1)^{\delta_{\nu j}} \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x')$$

## ⇒ TIME REVERSAL (T)

$$\hookrightarrow (t, \bar{x}) \xrightarrow{T} (-t, \bar{x}) \quad a_T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\bar{p} \xrightarrow{T} -\bar{p} \quad \begin{array}{c} \curvearrowright \bar{p} \\ \downarrow \end{array} \iff \begin{array}{c} \leftarrow -\bar{p} \\ \uparrow \end{array}$$

$$\bar{L} = \bar{x} \times \bar{p} \xrightarrow{T} -\bar{L}$$

$$\uparrow \xrightarrow{T} -\uparrow$$

## ↳ TRANSFORMATION OF FIELDS

- RECALL: FOR PROPER L.T. & FOR SPACE INVERSION FIELDS TRANSFORM THROUGH UNITARY TF

$$\Psi(x) \xrightarrow{a} U(a) \Psi(x) U^{-1}(a) = S^{-1}(a) \Psi(ax)$$

NOTE  $U(a)$ : UNITARY OPERATOR WHICH ACTS IN HILBERT SPACE

$S(a)$   $4 \times 4$  MATRIX WHICH ACTS IN DIRAC SPACE

SYMMETRY UNDER PROPER L.T. OR SPACE INVERSION REQUIRES MATRIX ELEMENTS TO BE INVARIANT

$$|\Psi_1\rangle \xrightarrow{a} U(a)|\Psi_1\rangle$$

$$|\Psi_2\rangle \xrightarrow{a} U(a)|\Psi_2\rangle$$

$$\langle \Psi_1 | \Psi_2 \rangle \xrightarrow{a} \langle U\Psi_1 | U\Psi_2 \rangle \stackrel{\text{SYMM.}}{=} \langle \Psi_1 | \Psi_2 \rangle$$

$$\downarrow \\ U^\dagger U = \mathbb{1} \quad (\text{UNITARY})$$

- FOR TIME-REVERSAL: INITIAL & FINAL STATES ARE REVERSED

$$|\Psi_1\rangle \xrightarrow{T} U(a_T) |\Psi_1\rangle$$

$$|\Psi_2\rangle \xrightarrow{T} U(a_T) |\Psi_2\rangle$$

$$\langle \Psi_1 | \Psi_2 \rangle \xrightarrow{T} \langle U\Psi_1 | U\Psi_2 \rangle$$

$$= \langle \Psi_2 | \Psi_1 \rangle = \langle \Psi_1 | \Psi_2 \rangle^*$$

T-REVERSAL  
INVARIANCE

REQUIRES AN ANTI-UNITARY OPERATOR

$$U^\dagger = U^{-1} \quad \text{AND} \quad \underline{\underline{U^\dagger(a_T) f(x) U(a_T) = f^*(x)}}$$

$$U^{-1}(a_T) = U(a_T)$$

$$(U^2(a_T) = 1)$$

↑  
COMPLEX FUNCTION

- $|\bar{p}, s\rangle \xrightarrow{T} |-\bar{p}, -s\rangle$

$$a(\bar{p}, s) \xrightarrow{T} U(a_T) a(\bar{p}, s) U(a_T) = a(-\bar{p}, -s)$$

$$b(\bar{p}, s) \xrightarrow{T} U(a_T) b(\bar{p}, s) U(a_T) = b(-\bar{p}, -s)$$

$$\bullet \quad \psi(x) \xrightarrow{T} U(a_T) \psi(x) U(a_T)$$

$$= \sum_s \int \frac{d^3 \bar{p}}{(2\pi)^3 \sqrt{2E_{\bar{p}}}} \left\{ a(-\bar{p}, -s) u^*(\bar{p}, s) e^{+ip \cdot x} + b^{\dagger}(-\bar{p}, -s) v^*(\bar{p}, s) e^{-ip \cdot x} \right\}$$

$$\downarrow \quad p' = (E_{\bar{p}}, -\bar{p})$$

$$p \cdot x = E_{\bar{p}} t - \bar{p} \cdot \bar{x} = -p' \cdot x'$$

$$\text{WITH } x' = (-t, \bar{x})$$

$$= \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}', s') u^*(-\bar{p}', -s') e^{-ip' \cdot x'} + b^{\dagger}(\bar{p}', s') v^*(-\bar{p}', -s') e^{+ip' \cdot x'} \right\}$$

$$\downarrow \quad u^*(-\bar{p}', -s') = \gamma^1 \gamma^3 u(\bar{p}', s')$$

$$= \gamma^1 \gamma^3 \sum_s \int \frac{d^3 \bar{p}'}{(2\pi)^3 \sqrt{2E_{\bar{p}'}}} \left\{ a(\bar{p}', s') u(\bar{p}', s') e^{-ip' \cdot x'} + b^{\dagger}(\bar{p}', s') v(\bar{p}', s') e^{+ip' \cdot x'} \right\}$$

$$\circ \circ \quad \underline{\underline{U(a_T) \psi(t, \bar{x}) U(a_T) = \gamma^1 \gamma^3 \psi(-t, \bar{x})}}$$

$$\hookrightarrow (\bar{\sigma} \cdot \bar{n}) \chi_s = \chi_s$$

$$\begin{aligned} (\bar{\sigma} \cdot \bar{n}) (-i\sigma^2 \chi_s^*) &= -i\sigma^2 (-\bar{\sigma} \cdot \bar{n})^* \chi_s^* \\ &= (-1) (-i\sigma^2 \chi_s^*) \end{aligned}$$

$$\therefore \underline{\underline{\chi_{-s} = -i\sigma^2 \chi_s^*}}$$

$$\hookrightarrow -\gamma^1 \gamma^3 (U(\bar{p}, s))^*$$

$$= \begin{pmatrix} -i\sigma^2 & \\ & -i\sigma^2 \end{pmatrix} \begin{pmatrix} \chi_s^* \\ \frac{\bar{\sigma} \cdot \bar{p}}{E_p + m} \chi_s^* \end{pmatrix} \sqrt{E_p + m}$$

$$= \sqrt{E_p + m} \begin{pmatrix} \chi_{-s} \\ -\frac{\bar{\sigma} \cdot \bar{p}}{E_p + m} \underbrace{(-i\sigma^2 \chi_s^*)}_{\chi_{-s}} \end{pmatrix}$$

$$= U(-\bar{p}, -s)$$

$$\therefore \left\| \begin{aligned} -\gamma^1 \gamma^3 (U(\bar{p}, s))^* &= U(-\bar{p}, -s) \\ -\gamma^1 \gamma^3 (U(\bar{p}, s))^* &= U(-\bar{p}, -s) \end{aligned} \right.$$

$$\rightsquigarrow (U(\bar{p}, s))^* = \gamma^1 \gamma^3 U(-\bar{p}, -s)$$

↳ TRANSFORMATION OF BILINEARS

$$U(a_T) \psi(x) U(a_T) = \gamma^1 \gamma^3 \psi(x') \quad \text{WITH } x' = (-t, \vec{x})$$

$$U(a_T) \bar{\psi}(x) U(a_T) = + \bar{\psi}(x') \gamma^3 \gamma^1$$

• SCALAR

$$\begin{aligned} \bar{\psi}(x) \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 \gamma^1 \gamma^3 \psi(x') \\ &= \bar{\psi}(x') \psi(x') \end{aligned}$$

• PSEUDO SCALAR (NOTE:  $i$  INCLUDED TO MAKE IT HERMITIAN!)

$$\begin{aligned} \bar{\psi}(x) i \gamma_5 \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 (-i \gamma_5) \gamma^1 \gamma^3 \psi(x') \\ &= -\bar{\psi}(x') i \gamma_5 \psi(x') \end{aligned}$$

• VECTOR

$$\begin{aligned} \bar{\psi}(x) \gamma^\mu \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 \gamma^{\mu*} \gamma^1 \gamma^3 \psi(x') \\ &= \begin{cases} \bar{\psi}(x') \gamma^\mu \psi(x') , & \mu = 0. \\ -\bar{\psi}(x') \gamma^\mu \psi(x') , & \mu = 1, 2, 3 \end{cases} \end{aligned}$$

• AXIAL-VECTOR

$$\begin{aligned} \bar{\psi}(x) \gamma^\mu \gamma_5 \psi(x) &\xrightarrow{T} \bar{\psi}(x') \gamma^3 \gamma^1 \gamma^{\mu*} \gamma_5 \gamma^1 \gamma^3 \psi(x') \\ &= \begin{cases} \bar{\psi}(x') \gamma^\mu \gamma_5 \psi(x') , & \mu = 0 \\ -\bar{\psi}(x') \gamma^\mu \gamma_5 \psi(x') , & \mu = 1, 2, 3 \end{cases} \end{aligned}$$

• TENSOR

$$\sigma^{\mu\nu} = \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$$

$$\bar{\Psi}(x) \sigma^{\mu\nu} \Psi(x) \xrightarrow{T} \bar{\Psi}(x') \gamma^3 \gamma^1 (\sigma^{\mu\nu})^* \gamma^1 \gamma^3 \Psi(x')$$

$$= \begin{cases} 0 & \mu=0, \nu=0 \\ + \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=0, \nu=i \\ + \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=i, \nu=0 \\ - \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x') & \mu=i, \nu=j \\ & i \neq j \end{cases}$$

$$= - (-1)^\mu (-1)^\nu \bar{\Psi}(x') \sigma^{\mu\nu} \Psi(x')$$



## ⇒ CHARGE CONJUGATION. (C)

↳ SYMMETRY UNDER PARTICLE ↔ ANTI-PARTICLE REVERSAL

$$a(\vec{p}, s) \xrightarrow{C} U(C) a(\vec{p}, s) U(C) = b(\vec{p}, s)$$

$$b(\vec{p}, s) \xrightarrow{C} U(C) b(\vec{p}, s) U(C) = a(\vec{p}, s)$$

WITH  $U(C) = U^{-1}(C)$  UNITARY OPERATOR

↳ RELATION BETWEEN U &  $\psi$  SPINORS

$$\begin{cases} \psi(\vec{p}, s) = +i \gamma^2 (U(\vec{p}, s))^* \\ U(\vec{p}, s) = +i \gamma^2 (\psi(\vec{p}, s))^* \end{cases}$$

PROOF

$$+i \gamma^2 (U(\vec{p}, s))^* = \sqrt{E_{\vec{p}+m}} \begin{pmatrix} +i \sigma^2 \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}+m}} \chi_s^* \\ -i \sigma^2 \chi_s^* \end{pmatrix}$$

$$= \sqrt{E_{\vec{p}+m}} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}+m}} (-i \sigma^2 \chi_s^*) \\ (-i \sigma^2 \chi_s^*) \end{pmatrix}$$

$$\downarrow \quad -i \sigma^2 \chi_s^* = \chi_{-s}$$

$$= \sqrt{E_{\vec{p}+m}} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E_{\vec{p}+m}} \chi_{-s} \\ \chi_{-s} \end{pmatrix} = \psi(\vec{p}, s)$$

↳ TRANSFORMATION OF FIELDS

$$\psi(x) \xrightarrow{C} U(C) \psi(x) U(C)$$

$$= \sum_{\lambda} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ b(\vec{p}, \lambda) u(\vec{p}, \lambda) e^{-i\vec{p} \cdot \vec{x}} + a^{\dagger}(\vec{p}, \lambda) v(\vec{p}, \lambda) e^{+i\vec{p} \cdot \vec{x}} \right\}$$

$$= i \gamma^2 \sum_{\lambda} \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left\{ b(\vec{p}, \lambda) v^*(\vec{p}, \lambda) e^{-i\vec{p} \cdot \vec{x}} + a^{\dagger}(\vec{p}, \lambda) u^*(\vec{p}, \lambda) e^{+i\vec{p} \cdot \vec{x}} \right\}$$

$$\underline{\underline{U(C) \psi(x) U(C) = i \gamma^2 \psi^*(x) = \left( \bar{\psi} \ i \gamma^0 \gamma^2 \right)^T}}$$

$$\bar{\psi}(x) \xrightarrow{C} U(C) \bar{\psi}(x) U(C)$$

$$= \left( U(C) \psi(x) U(C) \right)^{\dagger} \gamma^0$$

$$= \left( i \gamma^2 \psi^* \right)^{\dagger} \gamma^0$$

$$= \psi^T (-i) \gamma^0 \gamma^2$$

$$= \left( i \gamma^0 \gamma^2 \psi \right)^T$$

## ↳ TRANSFORMATION OF BILINEARS

### • SCALAR

$$\bar{\psi}(x) \psi(x) \xrightarrow{C} (i\gamma^0\gamma^2\psi)^T (\bar{\psi}i\gamma^0\gamma^2)^T$$

$$= - (\bar{\psi}i\gamma^0\gamma^2) (i\gamma^0\gamma^2\psi)$$

↑  
MINUS SIGN DUE TO  
FERMION ANTI-COMMUTATION

$$= + \bar{\psi}(x) \psi(x)$$

### • PSEUDO-SCALAR

$$\bar{\psi}(x) i\gamma_5 \psi(x) \xrightarrow{C} (i\gamma^0\gamma^2\psi)^T i\gamma_5 (\bar{\psi}i\gamma^0\gamma^2)^T$$

$$= - \bar{\psi}i\gamma^0\gamma^2 (i\gamma_5)^T i\gamma^0\gamma^2\psi$$

$$= \bar{\psi}i\gamma_5\psi$$

### • VECTOR

$$\bar{\psi}(x) \gamma^\mu \psi(x) \xrightarrow{C} (i\gamma^0\gamma^2\psi)^T \gamma^\mu (\bar{\psi}i\gamma^0\gamma^2)^T$$

$$= - \bar{\psi}i\gamma^0\gamma^2 \gamma^{\mu T} i\gamma^0\gamma^2\psi$$

$$\downarrow \gamma^{0T} = \gamma^0, \gamma^{2T} = \gamma^2, \gamma^{1T} = -\gamma^1, \gamma^{3T} = -\gamma^3$$

$$= - \bar{\psi}(x) \gamma^\mu \psi(x)$$

## • AXIAL - VECTOR

$$\begin{aligned}
\bar{\Psi}(x) \gamma^{\mu} \gamma_5 \Psi(x) &\xrightarrow{C} (i \gamma^0 \gamma^2 \Psi)^T \gamma^{\mu} \gamma_5 (\bar{\Psi} i \gamma^0 \gamma^2)^T \\
&= - \bar{\Psi} i \gamma^0 \gamma^2 \gamma_5 \gamma^{\mu T} i \gamma^0 \gamma^2 \Psi \\
&= - \bar{\Psi} \gamma^0 \gamma^2 \gamma^{\mu T} \gamma^0 \gamma^2 \gamma_5 \Psi \\
&= + \bar{\Psi}(x) \gamma^{\mu} \gamma_5 \Psi(x)
\end{aligned}$$

## • TENSOR

$$\begin{aligned}
\bar{\Psi} \sigma^{\mu\nu} \Psi &\xrightarrow{C} (i \gamma^0 \gamma^2 \Psi)^T \sigma^{\mu\nu} (\bar{\Psi} i \gamma^0 \gamma^2)^T \\
&= - \bar{\Psi} i \gamma^0 \gamma^2 (\sigma^{\mu\nu})^T i \gamma^0 \gamma^2 \Psi \\
&= - \bar{\Psi}(x) \sigma^{\mu\nu} \Psi(x)
\end{aligned}$$

⇒ CPT

	$\bar{\Psi}\Psi$	$\bar{\Psi}i\gamma_5\Psi$	$\bar{\Psi}\gamma^\mu\Psi$	$\bar{\Psi}\gamma^4\gamma_5\Psi$	$\bar{\Psi}\sigma^{\mu\nu}\Psi$
P	+1	-1	$(-1)^{\delta_{\mu i}}$	$-(-1)^{\delta_{\mu i}}$	$(-1)^{\delta_{\mu i}}(-1)^{\delta_{\nu j}}$
T	+1	-1	$(-1)^{\delta_{\mu i}}$	$(-1)^{\delta_{\mu i}}$	$-(-1)^{\delta_{\mu i}}(-1)^{\delta_{\nu j}}$
C	+1	+1	-1	+1	-1
CPT	+1	+1	-1	-1	+1

NOTE

$\partial_\mu \xrightarrow{P} (-1)^{\delta_{\mu i}} \partial_\mu$   
 $\partial_\mu \xrightarrow{T} -(-1)^{\delta_{\mu i}} \partial_\mu$   
 $\partial_\mu \xrightarrow{C} \partial_\mu$   
 $\partial_\mu \xrightarrow{CPT} -\partial_\mu$

$\mathcal{L}_{\text{DIRAC}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi$  IS INVARIANT UNDER P, T, C

⇒ PROOF OF INVARIANCE OF  $\mathcal{L}_{\text{DIRAC}} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi$   
 UNDER CPT

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• PARITY P

$$\bar{\Psi} \Psi \xrightarrow{P} \bar{\Psi} \Psi \quad \checkmark$$

$$\partial_\mu = \left( \frac{\partial}{\partial t}, \vec{\nabla} \right) \xrightarrow{P} \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right) = (-1)^\mu \partial_\mu$$

$$\bar{\Psi} \gamma^\mu \partial_\mu \Psi \xrightarrow{P} (-1)^\mu (-1)^\mu \bar{\Psi} \gamma^\mu \partial_\mu \Psi = \bar{\Psi} \gamma^\mu \partial_\mu \Psi \quad \checkmark$$

• TIME - REVERSAL T

$$\bar{\Psi} \Psi \xrightarrow{T} \bar{\Psi} \Psi \quad \checkmark$$

$$\partial_\mu = \left( \frac{\partial}{\partial t}, \vec{\nabla} \right) \xrightarrow{T} \left( -\frac{\partial}{\partial t}, \vec{\nabla} \right) = -(-1)^\mu \partial_\mu$$

$$\bar{\Psi} i \gamma^\mu \partial_\mu \Psi \xrightarrow{T} \bar{\Psi} \gamma^3 \gamma^1 (-i) \gamma^{\mu*} \gamma^1 \gamma^3 (\partial_\mu \Psi) (-1)(-1)^\mu$$

$$= i (-1)^\mu \underbrace{\bar{\Psi} \gamma^3 \gamma^1 \gamma^{\mu*} \gamma^1 \gamma^3 (\partial_\mu \Psi)}_{(-1)^\mu \bar{\Psi} \gamma^\mu (\partial_\mu \Psi)}$$

$$(-1)^\mu \bar{\Psi} \gamma^\mu (\partial_\mu \Psi)$$

$$= \bar{\Psi} i \gamma^\mu \partial_\mu \Psi \quad \checkmark$$

• CHARGE CONJUGATION C

$$\bar{\psi} \psi \xrightarrow{C} \bar{\psi} \psi \quad \checkmark \quad \partial_\mu \xrightarrow{C} \partial_\mu$$

$$\bar{\psi} i \gamma^\mu \partial_\mu \psi \xrightarrow{C} (i \gamma^0 \gamma^2 \psi)^T i \gamma^\mu \partial_\mu (\bar{\psi} i \gamma^0 \gamma^2)^T$$

$$= (-1) \left( \partial_\mu \bar{\psi} \right) i \gamma^0 \gamma^2 i \gamma^{\mu T} i \gamma^0 \gamma^2 \psi$$

↑  
FERMION  
ANTI-  
COMMUTATION.

$$= i (\partial_\mu \bar{\psi}) \gamma^0 \gamma^2 \gamma^{\mu T} \gamma^0 \gamma^2 \psi$$

$$\downarrow \quad \gamma^{0T} = \gamma^0, \quad \gamma^{2T} = \gamma^2 \\ \gamma^{1T} = -\gamma^1, \quad \gamma^{3T} = -\gamma^3$$

$$= -i (\partial_\mu \bar{\psi}) \gamma^\mu \psi$$

$$= i \bar{\psi} \gamma^\mu (\partial_\mu \psi) - i \underbrace{\partial_\mu (\bar{\psi} \gamma^\mu \psi)}_{\text{TOTAL DERIVATIVE}}$$

✓

TOTAL  
DERIVATIVE.