Theoretical Physics 6a (QFT): SS 2020
Exercise sheet 5

18.05.2020

Exercise 1. (40 points) : Dirac bilinears

Since a spinor turns into minus itself after a rotation over 2π, physical quantities must be bilinears in ψ, so that physical quantities turn into themselves after a rotation over 2π. These bilinears have the general form $\bar{\psi}\Gamma\psi$. There are 16 independent covariant ones related to 16 complex 4×4 matrices:

- $\Gamma_S = \mathbb{1}$ (scalar);
- $\Gamma_P = \gamma_5$ (pseudoscalar);
- $\Gamma_V^\mu = \gamma^\mu$ (vector);
- $\Gamma_A^\mu = \gamma^\mu\gamma_5$ (axial vector);
- $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ (tensor).

Without referring to any explicit representation for the Γ matrices,

(a)(10 points) show that $\Gamma^2 = \pm\mathbb{1}$.

(b)(10 points) show that for any Γ except Γ_S, we have $\mathrm{Tr}[\Gamma] = 0$. *Hint*: show first that for any Γ except Γ_S, there always exists a Γ' such that $\{\Gamma, \Gamma'\} = 0$.

(c)(10 points) check that the product of Γ_A^μ with any Γ is proportional to some Γ different from Γ_S;

(d)(10 points) and using the Lorentz transformation of the Dirac spinor $\psi'(x') = S(a)\psi(x)$ with $x'^\mu = a^\mu{}_\nu x^\nu$, check that the bilinears transform according to their name, *i.e.* $\bar{\psi}'\psi' = \bar{\psi}\psi$, $\bar{\psi}'\gamma_5\psi' = \det(a)\bar{\psi}\gamma_5\psi$, $\bar{\psi}'\gamma^\mu\psi' = a^\mu{}_\nu\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}'\gamma^\mu\gamma_5\psi' = \det(a)a^\mu{}_\nu\bar{\psi}\gamma^\mu\gamma_5\psi$ and $\bar{\psi}'\sigma^{\mu\nu}\psi' = a^\mu{}_\rho a^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi$.

Exercise 2. (60 points) : The angular momentum operator

(a)(20 points) Starting from the transformation law for the classical Dirac field under Lorentz transformations show that the generators of these transformations are given by

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}$$

(b)(20 points) The angular momentum of the Dirac field is

$$M_{\mu\nu} = \int d^3x\, \psi^\dagger(x) \left[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} \right] \psi(x)$$

Prove that

$$[M_{\mu\nu}, \psi(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\psi(x) - \frac{1}{2}\sigma_{\mu\nu}\psi(x)$$

(c)(20 points) Insert the normal mode expansions and construct explicitly the angular momentum tensor in terms of particles and anti-particles.