Theoretical Physics 6a (QFT): SS 2020
Exercise sheet 4

11.05.2020

Exercise 1. (30 points) : Axial current

For a Dirac field, the transformations

$$\psi(x) \to \psi'(x) = e^{i\alpha\gamma_5}\psi(x), \qquad \psi^\dagger(x) \to \psi^{\dagger\prime}(x) = \psi^\dagger(x)e^{-i\alpha\gamma_5},$$

where α is here an arbitrary real parameter, are called chiral phase transformations.

(a)(15 points) Show that the Dirac Lagrangian density $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ is invariant under chiral phase transformations in the zero-mass limit $m = 0$ only, and that the corresponding conserved current in this limit is the axial vector current $J_A^\mu \equiv \bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$.

(b)(15 points) Deduce the equations of motion for the fields

$$\psi_L(x) \equiv \frac{1}{2}(\mathbb{1} - \gamma_5)\psi(x), \qquad \psi_R(x) \equiv \frac{1}{2}(\mathbb{1} + \gamma_5)\psi(x),$$

for non-vanishing mass, and show that they decouple in the limit $m = 0$.

Hence, the Lagrangian density $\mathcal{L} = i\bar{\psi}_L\partial\!\!\!/\psi_L$ describes massless fermions with negative helicity and massless anti-fermions with positive helicity only. This field is called the Weyl field and can be used to describe the neutrinos as far as the latter can be considered as massless.

Exercise 2 (20 points): Dirac Propagator

Evaluate the spacelike Dirac correlator, i.e.

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \big(i\partial\!\!\!/ + m\big)_{\alpha\beta}\int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}e^{-ik(x-y)}$$

with $(x-y)^2 < 0$, explicitly. *Hint*: Use the result of Exercise 3 on Sheet 3.

Exercise 3 (50 points): Majorana Particles

The Majorana equation

$$i\bar{\sigma}^\mu \partial_\mu \chi - im\sigma^2\chi^* = 0$$

describes a massless 2-component fermion field ($\chi_a, a = 1, 2$) that transforms as the upper two components of a Dirac spinor.

(a)(20 points) Show that the action

$$S = \int \mathrm{d}^4x \left[\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \frac{im}{2} \left(\chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^* \right) \right]$$

is real and that varying S with respect to χ^* yields the Majorana equation. *Hint*: The field takes Grassmann numbers (i.e. anti-commuting variables) as values. The complex conjugate of a product of Grassmann numbers is calculated as

$$(\alpha\beta)^* = \beta^*\alpha^*.$$

(b)(15 points) Let us write a 4-component Dirac field in terms of 2-component spinors as

$$\psi(x) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \chi_1(x) \\ i\sigma^2\chi_2^*(x) \end{pmatrix}.$$

Rewrite the Dirac Lagrangian in terms of χ_1 and χ_2.

(c)(15 points) Calculate the divergencies of the currents

$$J_a^\mu = \chi^\dagger \bar{\sigma}^\mu \chi, \quad J_b^\mu = \chi_1^\dagger \bar{\sigma}^\mu \chi_1 - \chi_2^\dagger \bar{\sigma}^\mu \chi_2.$$

Express your result in a way that the only Pauli matrix it contains is σ^2.