Theoretical Physics 6a (QFT): SS 2020
Exercise sheet 3

04.05.2020

Exercise 1 (30 points) : Dirac Field

The Free Dirac Lagrangian is given by:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi,$$

where the normal mode expansion for the fields are:

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\sum_s \left\{a(\vec{k},s)u(\vec{k},s)e^{-ikx} + b^\dagger(\vec{k},s)v(\vec{k},s)e^{ikx}\right\}$$

$$\bar{\psi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\sum_s \left\{b(\vec{k},s)\bar{v}(\vec{k},s)e^{-ikx} + a^\dagger(\vec{k},s)\bar{u}(\vec{k},s)e^{ikx}\right\}.$$

(a)(10 points) Show, that the momentum operator

$$\vec{P} = \int d^3\vec{x}\ \psi^\dagger(x)(-i\vec{\nabla})\psi(x)$$

can be expressed as:

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3}\sum_s \vec{k}\left\{a^\dagger(\vec{k},s)a(\vec{k},s) + b^\dagger(\vec{k},s)b(\vec{k},s)\right\}.$$

(b)(10 points) Show, that the conserved charge

$$Q = \int d^3\vec{x}\ \bar{\psi}(x)\gamma_0\psi(x)$$

can be expressed as:

$$Q = \int \frac{d^3\vec{k}}{(2\pi)^3}\sum_s \left\{a^\dagger(\vec{k},s)a(\vec{k},s) - b^\dagger(\vec{k},s)b(\vec{k},s)\right\},$$

and explain the meaning of the relative minus sign in the expression above.

(c)(10 points) Show, that if the Dirac field were quantized according to the Bose-Einstein statistics, that is, through commutators as for the Klein-Gordon field, one would get the following Hamiltonian:

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\sum_s E_{\vec{k}}\left\{a^\dagger(\vec{k},s)a(\vec{k},s) - b^\dagger(\vec{k},s)b(\vec{k},s)\right\}$$

and explain why this would lead to unphysical results for the energy spectrum.

Exercise 2. (50 points) : Dirac matrix calculus

Without using an explicit representation of the gamma matrices, show the following identities:

$$\begin{aligned}
\gamma_\mu \gamma^\mu &= 4, \\
\mathrm{Tr}[\gamma^\mu \gamma^\nu] &= 4g^{\mu\nu}, \\
\mathrm{Tr}\left[\not{a}\not{b}\not{c}\not{d}\right] &= 4(a \cdot b\ c \cdot d - a \cdot c\ b \cdot d + a \cdot d\ b \cdot c), \\
\gamma_5 &= \frac{i}{4!}\epsilon_{\mu\nu\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta, \\
\gamma_5^2 &= 1, \\
[\gamma_5, \gamma^\mu]_+ &= 0, \\
\mathrm{Tr}[\gamma_5] &= 0, \\
\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma_5] &= 0, \\
\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma_5\right] &= 4i\epsilon^{\mu\nu\alpha\beta}, \\
\mathrm{Tr}[\gamma^{\mu_1}...\gamma^{\mu_n}] &= 0, \text{ if } n \text{ is odd},
\end{aligned}$$

where $\not{a} \equiv a^\mu \gamma_\mu$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\epsilon_{0123} = 1$

Exercise 3 (20 points): Klein-Gordon Propagator

Evaluate the spacelike Klein-Gordon correlator, i.e.

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3}\frac{1}{2E_{\vec{k}}}e^{-ik(x-y)}$$

with $(x-y)^2 < 0$, explicitly and express it in terms of a modified Bessel function of second kind

$$K_n(z) \equiv \frac{\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\left(\frac{z}{2}\right)^n \int_1^\infty \mathrm{d}\rho\, e^{-\rho z}(\rho^2-1)^{n-1/2}\,.$$