Lecture 9

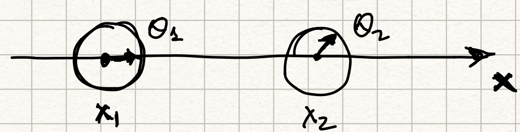
III The Photon Field

1) Abelian Gauge Theory, QED

Local phase transformation

$$\mathcal{L}_{DIRAC} = \bar{\psi}(x)\left(i\gamma^\mu \partial_\mu - m\right)\psi(x)$$

Interaction ? free field



$\theta_1 = 0$
$\theta_2 = \theta$

$$\rightarrow \quad \psi(x) \xrightarrow{U(1)} e^{i\theta(x)}\psi(x) \qquad U(1) \text{ local}$$

$$\bar{\psi}(x) \rightarrow e^{-i\theta(x)}\bar{\psi}(x)$$

$$\bar{\psi}\psi \rightarrow \underbrace{e^{i\theta}e^{-i\theta}}_{1}\bar{\psi}\psi = \bar{\psi}\psi$$

$$\partial_\mu \psi(x) \rightarrow \partial_\mu\left(e^{i\theta(x)}\psi(x)\right) = e^{i\theta(x)}\left(\partial_\mu\psi(x) + i\underbrace{(\partial_\mu\theta(x))\psi(x)}_{\text{extra term}}\right)$$

$$\mathcal{L}_{DIRAC} \xrightarrow{U(1)} \mathcal{L}_{DIRAC} - (\partial_\mu\theta)\,\bar{\psi}\gamma^\mu\psi$$

$\mathcal{L}_{DIRAC}$ is not invariant under local $U(1)$

Introduce covariant derivative

$$\partial_\mu\psi \text{ replace } D_\mu\psi \stackrel{\text{def}}{=} \partial_\mu\psi + ieA_\mu\psi$$

Request $$D_\mu\psi \xrightarrow{U(1)} e^{i\theta(x)}D_\mu\psi$$

$$A_\mu \rightarrow A'_\mu = ?$$

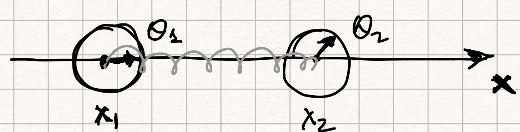
$$D_\mu \psi \to e^{i\theta(x)} \underline{D_\mu \psi} = e^{i\theta(x)} (\partial_\mu \psi + ieA_\mu \psi)$$

$$D_\mu \psi = \partial_\mu \psi + ie A_\mu \psi \to \underbrace{\partial_\mu (e^{i\theta(x)} \psi)} + ieA'_\mu \cdot e^{i\theta(x)} \psi$$

$$e^{i\theta} \partial_\mu \psi + e^{i\theta} i \partial_\mu \theta \, \psi$$

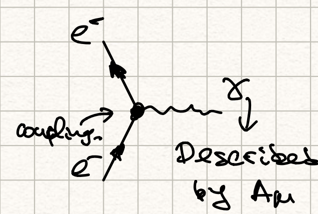
$$e A_\mu = e A'_\mu + \partial_\mu \theta$$

$$A_\mu \xrightarrow{U(1)} A'_\mu = A_\mu - 1/e \, \partial_\mu \theta$$



$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi =$$

$$= \mathcal{L}_{DIRAC} - \underbrace{e(\bar{\psi}\gamma^\mu \psi) A_\mu}_{\underline{J^\mu_{e.m.} \cdot A_\mu}}$$



$$\psi(x) \to e^{i\theta(x)} \psi$$

$$A_\mu \to A_\mu - 1/e \, \partial_\mu \theta$$

$$\mathcal{L}_{QED} = \mathcal{L}_{DIRAC} + \mathcal{L}_{INT} + \mathcal{L}_{\underline{PHOTON}}$$

$\mathcal{L}_{DIRAC}$ ↓ free fermion

$\mathcal{L}_{PHOTON}$ ↓ free photon field

QED = Quantum Electro Dynamics

Euler, Lagrange eq ⟶ Maxwell eqs.

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{j} \longrightarrow \text{antisym. field tensor} \quad F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$
$$(\vec{\nabla}\cdot\vec{E}) = \rho$$
$$(\vec{\nabla}\cdot\vec{B}) = 0$$

$\longrightarrow$ Covariant form

$$\partial_\mu F^{\mu\nu} = J^\nu_{em} \; ; \quad J^\nu_{em} = (\rho, \vec{j})$$
$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \, , \quad \tilde{F}_{\mu\nu} \equiv \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$$

4-vector potential $A^\mu = (\varphi, \vec{A})$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\mathcal{L}_{INT} + \mathcal{L}_{PHOTON} = -J^\nu_{em} A_\nu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\frac{\partial\mathcal{L}}{\partial\varphi_\eta} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_\eta)} = 0 \rightarrow \frac{\partial\mathcal{L}}{\partial A_\nu} = -J^\nu_{em}$$

$$\varphi_\eta \equiv A_\nu \qquad \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

$$\frac{\partial\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\left(\frac{\partial F_{\mu\nu}}{\partial(\partial_\mu A_\nu)} F^{\mu\nu} + F^{\mu\nu} \frac{\partial F_{\mu\nu}}{\partial(\partial_\mu A_\nu)}\right)$$
$$= -\frac{1}{2}\left(\frac{\partial F_{\mu\nu}}{\partial(\partial_\mu A_\nu)} F^{\mu\nu}\right) = -F^{\mu\nu} \qquad -F^{\mu\nu}$$

$$\frac{\partial(\partial_\mu A_\nu - \partial_\nu A_\mu)}{\partial(\partial_\mu A_\nu)} \cdot F^{\mu\nu} = \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\mu A_\nu) F^{\mu\nu} - \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\mu A_\nu) F^{\nu\mu}$$
$$= 2 \cdot F^{\mu\nu}$$

from E.L. eq. $\quad -J^\nu_{em} + \partial_\mu F^{\mu\nu} = 0$

$$\partial_\mu F^{\mu\nu} = J^\nu_{em}$$ ← inhom. Maxwell eq. #

$$F_{\mu\nu} \xrightarrow{U(1)} \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu (A_\nu - 1/e\, \cancel{\partial_\nu \theta}) - \partial_\nu (A_\mu - 1/e\, \cancel{\partial_\mu \theta})$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu = \underline{F_{\mu\nu}}$$

$$A^\mu \to \underline{A'^\mu = A^\mu - 1/e\, \partial^\mu \theta}$$

$\mathcal{L}_{PHOTON} \xrightarrow{U(1)} \mathcal{L}_{PHOTON}$ ⇒ need to fix redundancy

Gauge choice

1) $\partial_\mu A^\mu = 0$ Lorenz gauge (Lorentz covariant)

2) $\vec{\nabla} \vec{A} = 0$, $A^0 = 0$ Coulomb gauge

$$\partial_\mu A^\mu \to \partial_\mu A'^\mu = \partial_\mu A^\mu - 1/e\, \Box \theta$$

$$\Box \theta = 0$$

2) Quantization of Photon field

- Do not impose gauge

issue 1: Conjugate momenta

$$\mathcal{L}_{\text{PHOTON}} = -\frac{1}{4} \overbrace{F_{\mu\nu}}^{\partial_\mu A_\nu - \partial_\nu A_\mu} F^{\mu\nu}$$

$$\pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu}$$

$\partial_\mu A_\nu$

$$\dot{A}_\nu = \frac{\partial A_\nu}{\partial t}$$

$$\pi^0 = 0$$

$$\pi^i = -F^{0i} = -E_i$$

issue 2: Photon propagator

(1) In the Klein-Gordon eq. (classical)

$$(\partial_\mu \partial^\mu + m^2)\,\varphi(x) = 0$$

$$\varphi(x) \sim e^{-ikx}$$

$$\underbrace{(-k^2 + m^2)}\,\varphi(x) = 0$$

$$(\partial_\mu \partial^\mu + m^2)\,\Delta_F(x) = -\delta^{(4)}(x)$$

$$\Delta_F(k) = \frac{1}{k^2 - m^2}$$

(2) Free photon field for A^ν

$$\partial_\mu F^{\mu\nu} = 0$$

$$\partial_{\mu'} \partial^{\mu'} A^\nu - \partial_\mu \partial^\nu A^\mu = 0 \qquad A_\mu \sim e^{-ikx} \cdot \epsilon_\mu$$

$$\downarrow$$

$$\partial_{\mu'} \partial^{\mu'} g^{\nu\mu} A_\mu - \partial^\nu \partial^\mu A_\mu = 0$$

$$\underbrace{(-g^{\mu\nu} k^2 + k^\nu k^\mu)}_{\text{inverse of this operator?}} A_\mu = 0$$

$$(-k^2 g^{\mu\nu} + k^\mu k^\nu)(\underbrace{A\, g_{\nu\lambda} + B\, k_\nu k_\lambda}_{}) = g^\mu_\lambda$$

$$\underline{-A k^2}\, g^\mu_\lambda + \underline{A}\, k^\mu k_\lambda - \cancel{k^2 B k^\mu k_\lambda} + \cancel{k^2 B k^\mu k_\lambda} = \underline{g^\mu_\lambda}$$

$$\begin{cases} -A k^2 = 1 \\ A = 0 \end{cases}$$

impossible

<u>no inverse</u>

Gauge: $\partial_\mu A^\mu = 0$ (classical level)

M. eq. $\partial_\mu \partial^\mu A^\nu - \partial^\nu(\cancelto{0}{\partial_\mu A^\mu}) = 0$

$\Box A^\nu = 0$ (Maxwell eq. for Lorenz gauge)

Gauge fixing term in the Lagrangian

$$\mathcal{L}_{\text{photon}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \underbrace{\frac{1}{2}(\partial_\mu \hat{A}^\mu)^2}_{\text{gauge fixing term}}$$

E.L. eq.

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} - (\partial_\alpha A^\alpha) g^{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

$$\partial_\mu F^{\mu\nu} + \partial^\nu(\partial_\alpha A^\alpha) = 0$$

$$\Box A^\nu - \partial^\nu(\cancel{\partial_\mu A^\mu}) + \partial^\nu(\cancel{\partial_\alpha A^\alpha}) = 0$$

$$\Box A^\nu = 0$$

Solution to issue 2.

$$\Box A^\nu = 0$$

$$(-k^2 g^{\mu\nu}) A_\nu = 0$$

$$(-k^2 g^{\mu\nu})\left(\underline{A\, g_{\nu\lambda}} + B\, k_\nu k_\lambda \right) = g^{\mu}_{\lambda}$$

$$A = -\frac{1}{k^2} \; ; \; B = 0$$

Photon prop. in Lorenz gauge $\sim \dfrac{g^{\mu\nu}}{k^2 + i\epsilon}$

solution to issue 1

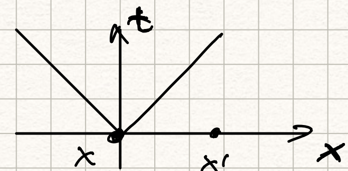
$\mathcal{L}_{\text{gauge fixing}} = \frac{1}{2} (\partial_\mu \hat{A}^\mu)^2$

$$\pi^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu} - (\partial_\alpha \hat{A}^\alpha)\, g^{0\nu}$$

$\pi^0 = -\partial_\alpha A^\alpha$ is ok; $\partial_\alpha \hat{A}^\alpha$

$$\pi^i = -F^{0i} = -\partial^0 A^i + \partial^i A^0$$

ETCR postulate



$$\left| \begin{array}{l} [A^\mu(\vec{x}, t), A^\nu(\vec{x}', t)] = 0 \\ [\pi^\mu(\vec{x}, t), \pi^\nu(\vec{x}', t)] = 0 \\ [A^\mu(\vec{x}, t), \pi^\nu(\vec{x}', t)] = i g^{\mu\nu} \delta^{(3)}(\vec{x} - \vec{x}') \end{array} \right.$$