Lecture 8 (13 May 2020)

II Dirac field

4) Symmetries of Dirac theory

Consider Lorentz trasf.

$$x^\mu \to x'^\mu = a^\mu{}_\nu x^\nu$$

$$\psi(x) \to \psi'(x') = S(a)\psi(x)$$

$$S^{-1}(a) = S(a^{-1})$$

from Lorentz covariance of Dirac eq.

$$S^{-1}(a)\gamma^\nu S(a) = a^\nu{}_\mu \gamma^\mu$$

Example:

$$S(a) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right); \quad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

rotation around z axis over angle φ

$$\sigma^{\mu\nu} = -\sigma^{\nu\mu}$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$S(a) = \exp\left(-\frac{i}{2}\varphi\Sigma_3\right) \qquad \sigma^{12} = \begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} \equiv \Sigma_3$$

Consider spinor with spin proj +1/2 along z-axis

$$\psi(x); \quad \Sigma_3\psi = +\psi \qquad \vec{S} = \frac{1}{2}\vec{\Sigma}$$

$$\psi'(x') = S(a)\psi(x) = \exp\left(-\frac{i}{2}\varphi\Sigma_3\right)\psi(x)$$

$$= \exp\left(-\frac{i}{2}\varphi\right)\psi(x)$$

$$\varphi = 4\pi, \quad \exp(-2\pi i) = 1$$

$\Rightarrow$ The physical quantity must be bilinear in ψ
i.e. turn into itself
after rot. $\varphi = 2\pi$

$$\mathcal{L} = \bar{\psi}(i\underline{\gamma^\mu}\partial_\mu - \underline{m})\psi \; ; \quad \underline{\bar{\psi}\,1\,\psi}$$

$\underset{(1\times4)}{\bar\psi}\ \underset{(4\times4)}{\mathbb{1}}\ \underset{(4\times1)}{\psi}$: $\quad \bar\psi'(x')\psi'(x') \to \bar\psi(x)\psi(x) \quad$ scalar

$$\psi'(x') = S(a)\psi(x)\ ; \quad \bar\psi \equiv \psi^\dagger\gamma^0$$

$$\bar\psi'(x')\psi'(x') = \left(S\psi(x)\right)^\dagger\gamma^0\left(S\psi(x)\right) =$$

$$= \psi^\dagger(x)\underbrace{S^\dagger\gamma^0}_{\gamma^0 S^{-1}} S\,\psi(x) = \bar\psi(x)\psi(x)$$

$$S^\dagger\gamma^0 = \left(\exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\right)^\dagger\gamma^0 = \exp\left(\frac{i}{4}\omega_{\mu\nu}\underbrace{\sigma^{\mu\nu\dagger}}_{\gamma^0\sigma^{\mu\nu}\gamma^0}\right)\gamma^0$$

$(\sigma^{\mu\nu})^\dagger = \gamma^0\sigma^{\mu\nu}\gamma^0$

$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$

$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

$$= \gamma^0 + \frac{i}{4}\omega_{\mu\nu}\gamma^0\sigma^{\mu\nu}\underbrace{\gamma^0\gamma^0}_{1}$$

$$= \gamma^0\underbrace{\exp\left(+\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)}_{S^{-1}} = \gamma^0 S^{-1}$$

Transformation of field operators in Hilbert space under Lorentz transf.

$\hat\psi(x)$; $\quad |\vec p, s\rangle = \sqrt{2E_p}\, a^\dagger(p,s)|0\rangle$

$x'^\mu = a^\mu{}_\nu x^\nu \longrightarrow a$

$$|\vec p, s\rangle \to |\vec p', s'\rangle = U(a)|\vec p, s\rangle$$

Norm is invar.

$$\langle \vec p, s|\vec p, s\rangle = \langle \vec p', s'|\vec p', s'\rangle = \langle \vec p, s|\underbrace{U^\dagger U}_{\mathbb{1}}|\vec p, s\rangle$$

$U^\dagger U = 1$ unitary trs

act on spinors

$$\hat\psi(x) \to U(a)\hat\psi(x)U^\dagger(a) = S(a)\hat\psi(a^{-1}x) \quad \text{passive L.T}$$

act on $\hat a, \hat b$

$$S^{-1}(a)\hat\psi(ax) \quad \text{active L.T.}$$

$a \leftrightarrow a^{-1}$

- PARITY

$$(t, \vec{x}) \xrightarrow{P} (t, -\vec{x}) \qquad a^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \equiv a_P$$

$$\vec{p} \xrightarrow{P} -\vec{p} \qquad \det(a) = -1$$

$$\cancel{g^{\mu\nu}}$$

Angular momentum

$$\vec{L} = \vec{x} \times \vec{p} \overset{P}{=} \vec{L} \qquad g^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

- Application of P on spinor $\psi(x)$

$$\psi'(x') = S(a_P)\,\psi(x) \qquad \psi''(x'') = S(a_P)\,\psi'(x') = S(a_P)^2\,\psi(x)$$

$$S(a_P)^2 = \mathbb{1}$$

$$S(a_P) = S^{-1}(a_P)$$

$$S^{-1}(a)\,\gamma^{\nu}\,S(a) = a^{\nu}{}_{\mu}\gamma^{\mu}$$

$$S^{-1}\gamma^0 S = \gamma^0$$

$$S^{-1}\gamma^i S = -\gamma^i \qquad i = 1,2,3$$

$$\Downarrow$$

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$$

$$\gamma^0\gamma^i + \gamma^i\gamma^0 = 0$$

$$S = S^{-1} = \gamma^0\eta, \quad \eta^2 = \mathbb{1}$$

Phase factor: $\eta = \pm 1$

- Transformation of field operator $\psi(x)$

$$\hat{\psi}(t, \vec{x}) \to U(a_P)\,\hat{\psi}(t, \vec{x})\,U^{+}(a_P) = \underline{\eta\gamma^0}\,\hat{\psi}(t, -\vec{x})$$

Proof:

$$\hat{\psi}(t, \vec{x}) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(\hat{a}(\vec{p}, s)\,u(\vec{p}, s)\,e^{-ipx} + \hat{b}^{+}(\vec{p}, s)\,v(\vec{p}, s)\,e^{+ipx} \right)$$

$$\hat{a}(\vec{p}, s) \xrightarrow{P} U(a_P)\,\hat{a}(\vec{p}, s)\,U^{+}(a_P) = \eta_a\,\hat{a}(-\vec{p}, s)$$

$$\hat{b}(\vec{p}, s) \xrightarrow{P} U(a_P)\,\hat{b}(\vec{p}, s)\,U^{+}(a_P) = \eta_b\,\hat{b}(-\vec{p}, s)$$

$$\eta_a^2 = 1, \quad \eta_b^2 = 1$$

$$\psi(t,\vec{x}) \xrightarrow{P} \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \Big(\eta_a\, a(-\vec{p},s)\, u(\vec{p},s)\, e^{-ipx} + \eta_b^*\, b^+(-\vec{p},s)\, v(\vec{p},s)\, e^{ipx} \Big) =$$

$\vec{p} \to \vec{p}\,' = -\vec{p}$

$$= \sum_s \int \frac{d^3\vec{p}\,'}{(2\pi)^3\sqrt{2E_{\vec{p}\,'}}} \Big(\eta_a\, a(\vec{p}\,',s)\, u(-\vec{p}\,',s)\, e^{-iE_{p'}t - i\vec{p}\,'\vec{x}} + \eta_b^*\, b^+(\vec{p}\,',s)\, v(-\vec{p}\,',s)\, e^{iE_{p'}t + i\vec{p}\,'\vec{x}} \Big)$$

$$u(\vec{p},s) = N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+m}\chi_s \end{pmatrix} \Rightarrow u(-\vec{p}\,',s) = N \begin{pmatrix} \chi_s \\ -\frac{\vec{\sigma}\vec{p}\,'}{E_{\vec{p}\,'}+m}\chi_s \end{pmatrix} = \gamma^0 u(\vec{p}\,',s)$$

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad v(-\vec{p}\,',s) = -\gamma^0 v(\vec{p}\,',s)$$

$$x^\mu \to x'^\mu = a^\mu{}_\nu x^\nu \Rightarrow x' \equiv (t, -\vec{x})$$

$$= \sum_s \int \frac{d^3\vec{p}\,'}{(2\pi)^3\sqrt{2E_{\vec{p}\,'}}} \Big(\eta_a\, a(\vec{p}\,',s)\, \gamma^0 u(\vec{p}\,',s)\, e^{-ip'x'} - \eta_b^*\, b^+(\vec{p}\,',s)\,\gamma^0 v(\vec{p}\,',s)\, e^{ip'x'} \Big) =$$

$$\underline{\hat{\psi}(t,\vec{x})} \xrightarrow{P} (\ldots.)\, \underline{\hat{\psi}(t,-\vec{x})}$$

$$\hat{\psi}(x') = \sum_s \int \frac{d^3\vec{p}\,'}{(2\pi)^3\sqrt{2E_{\vec{p}\,'}}} \Big(\hat{a}(\vec{p}\,',s)\, u(\vec{p}\,',s)\, e^{-ip'x'} + \hat{b}^+(\vec{p}\,',s)\, v(\vec{p}\,',s)\, e^{+ip'x'} \Big)$$

$$\Downarrow$$

$$\hat{\psi}(t,\vec{x}) \xrightarrow{P} \gamma^0 \eta_a\, \hat{\psi}(t,-\vec{x}) \qquad \eta_b^* = -\eta_a$$

particles and antipart should have opposite intrinsic parities

$\eta_a = +1$ convention

Transformation of bilinears: Parity

- scalar

$$\bar\psi(x)\psi(x) \xrightarrow{P} \underbrace{|\eta_a|^2}_{=\mathbb{1}}\,\bar\psi(x')\gamma^0\gamma^0\psi(x') = \bar\psi(x')\psi(x)$$

- pseudo scalar

$$\bar\psi(x)\,i\gamma_5\,\psi(x) \xrightarrow{P} \bar\psi(x')\gamma^0\, i\overbrace{\gamma_5\gamma^0}^{-\gamma^0\gamma_5}\psi(x') =$$

$$= -\bar\psi(x)\,i\gamma_5\,\psi(x)$$

$(\bar\psi i\gamma_5\psi)^\dagger = \bar\psi i\gamma_5\psi$

Lagrangian is hermitian

- vector

$$\underbrace{\bar\psi(x)\gamma^\mu\psi(x)}_{J^\mu} \xrightarrow{P} \bar\psi'(x')\gamma^0\gamma^\mu\gamma^0\psi(x') =$$

$$= \begin{cases} \bar\psi\gamma^0\psi, & \mu=0 \\ -\bar\psi\gamma^i\psi, & \mu=i \end{cases} = a^\mu{}_\nu\,\bar\psi\gamma^\nu\psi$$

or

$(-1)^{\delta_{\mu i}}\,\bar\psi\gamma^\mu\psi$

i = running all the number from 1...3

$$J'^\mu(x') = a^\mu{}_\nu J^\nu(x)$$

- pseudo vector (axial)

$$\bar\psi\gamma^\mu\gamma_5\psi \xrightarrow{P} \bar\psi\gamma^0\gamma^\mu\overbrace{\gamma_5\gamma^0}^{-\gamma^0\gamma_5}\psi =$$

$$= \begin{cases} -\bar\psi\gamma^0\psi & \mu=0 \\ \bar\psi\gamma^i\psi & \mu=i \end{cases}$$

- Tensor

$$\bar\psi\underbrace{\sigma^{\mu\nu}}_{\frac{i}{2}(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)}\psi \xrightarrow{P} \bar\psi\gamma^0\sigma^{\mu\nu}\gamma^0\psi =$$

$$= \begin{cases} 0 & \mu,\nu=0;\ \mu=\nu=i \\ -\bar\psi\sigma^{\mu\nu}\psi & \mu=0,\ \nu=i \\ -\bar\psi\sigma^{\mu\nu}\psi & \mu=i,\ \nu=0 \\ +\bar\psi\sigma^{\mu\nu}\psi & \mu=i,\ \nu=j,\ i\neq j \end{cases}$$

$$\gamma^0(\gamma^{\mu=0}\gamma^i - \gamma^i\gamma^{\mu=0})\gamma^0$$

$$-(\gamma^0\gamma^i - \gamma^i\gamma^0)$$

• Charge conjugation (C)

This is a symmetry under

particle $\longleftrightarrow$ antiparticle

$$a(\vec{p},s) \xrightarrow{C} U a(\vec{p},s) U^{\dagger} = b(\vec{p},s)$$

$$b(\vec{p},s) \xrightarrow{C} U b(\vec{p},s) U^{\dagger} = a(\vec{p},s)$$

phases = 1

$\eta_a = \eta_b = 1$

Transformation of the field operator

$$\psi(x) \longrightarrow \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left(b(\vec{p},s)\, u(\vec{p},s)\, e^{-ipx} + a^{\dagger}(\vec{p},s)\, v(\vec{p},s)\, e^{ipx} \right) =$$

$$\left| \begin{array}{l} u(\vec{p},s) = i\gamma^2 \left(v(\vec{p},s)\right)^* \\ v(\vec{p},s) = i\gamma^2 \left(u(\vec{p},s)\right)^* \end{array} \right. \quad ; \qquad \vec{\sigma}\sigma^2 = -\sigma^2 \vec{\sigma}^*$$

$$i\gamma^2 \left(u(\vec{p},s)\right)^* = N \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \chi_s^* \\ \frac{\vec{\sigma}^* \vec{p}}{E_{\vec{p}}+m} \chi_s^* \end{pmatrix} =$$

$$= N \begin{pmatrix} i\sigma_2 \frac{\vec{\sigma}^* \vec{p}}{E_{\vec{p}}+m} \chi_s^* \\ -i\sigma_2 \chi_s^* \end{pmatrix} = N \begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+m} \overbrace{(-i\sigma_2 \chi_s^*)}^{\chi_{-s}} \\ \underbrace{(-i\sigma_2 \chi_s^*)}_{\chi_{-s}} \end{pmatrix}$$

$$= v(\vec{p},s)$$

$$-i\sigma_2 \chi_s^* \overset{?}{=} \leftarrow \chi_{-s}$$

$$\left(\frac{\vec{\sigma}\vec{n}}{2}\right) \chi_s = s\, \chi_s \qquad s = \pm 1/2$$

$$\vec{\sigma}\vec{n}\,\chi_{+1/2} = \chi_{+1/2} \quad \rightarrow \quad \left((\vec{\sigma}\vec{n})\chi_{+1/2}\right)^* = \chi^*_{+1/2}$$

$$\vec{\sigma}\vec{n}\,\chi_{-1/2} = -\chi_{-1/2} \qquad (*)$$

Calculate the object

$$\vec{\sigma}\vec{n}\left(-i\sigma^2\chi^*_{+1/2}\right) = -\sigma^2\Big(-i\underbrace{(\vec{\sigma}^*\vec{n})\chi^*_{1/2}}_{\chi^*_{1/2}}\Big) =$$

$$\vec{\sigma}\sigma^2 = -\sigma^2\vec{\sigma}^*$$

$$= -\left(-i\sigma^2\chi^*_{1/2}\right) \qquad (*,*)$$

from $(*)$ and $(*,*)$ $\quad \chi_{-s} = -i\sigma^2\chi^*_s$

$$\psi(x) \xrightarrow{C} i\gamma^2\sum_s\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\Big(b(\vec{p},s)\,v^*(\vec{p},s)\,e^{-ipx} + a^\dagger(\vec{p},s)\,u^*(\vec{p},s)\,e^{ipx}\Big)$$

$$= i\gamma^2\,\psi^*(x)$$

$$\psi(x) \xrightarrow{C} i\gamma^2\psi^*(x) = \left(\bar{\psi}\,i\gamma^0\gamma^2\right)^T$$

$$\bar{\psi}(x) \xrightarrow{C} \left(i\gamma^2\psi^*(x)\right)^\dagger\gamma^0 = \psi^T(-i)(-\gamma^2)\gamma^0 =$$

$$= \left(i\gamma^0\gamma^2\psi\right)^T$$

$(\gamma^0)^T = \gamma^0 \qquad (\gamma^1)^T = -\gamma^1$

$(\gamma^2)^T = \gamma^2 \qquad (\gamma^3)^T = -\gamma^3$

Transformation of bilinears: charge conjugation

- scalar $\underbrace{\bar{\psi}\mathbb{1}\psi}_{\text{number}} \xrightarrow{C} \left((i\gamma^0\gamma^2\psi)^T(\bar{\psi}\,i\gamma^0\gamma^2)^T\right)^T =$

$(1\times4)\;(4\times1) = 1\times1$

$\wedge$

(4×4)

$$\overset{=-1}{i^2}\gamma^0\gamma^2\underbrace{\gamma^0\gamma^2}_{=-\gamma^0\gamma^2} = \overset{1}{(\gamma^0)^2}\underbrace{(\gamma^2)^2}_{-1} = -1$$

$$= -(\bar\psi i\gamma^0\gamma^2)(i\gamma^0\gamma^2\psi) = \bar\psi \mathbb{1}\psi$$

fermion anti-commut.

- pseudo scal

$$\bar\psi i\gamma_5\psi = \left((i\gamma^0\gamma^2\psi)^T i\gamma_5(\bar\psi i\gamma^0\gamma^2)^T\right)^T$$

$$= -\bar\psi i\gamma^0\gamma^2(i\gamma_5)^T i\gamma^0\gamma^2\psi$$

$$= \bar\psi i\gamma_5\psi$$

$\mathcal{L}_{DIRAC} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$ symmetric under **C, P, T, CPT**

- $\bar\psi\psi \xrightarrow{P,C} \bar\psi\psi$

- $\bar\psi i\gamma^\mu\partial_\mu\psi \xrightarrow{P} \underbrace{a^\mu{}_\nu\, a_\mu{}^\lambda}_{g^\lambda_\nu}\bar\psi i\gamma^\nu\partial_\lambda\psi = \bar\psi i\gamma^\nu\partial_\nu\psi$

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec\nabla\right)$$

- $\bar\psi i\gamma^\mu\partial_\mu\psi \xrightarrow{C} \bar\psi i\gamma^\mu\partial_\mu\psi$