Lecture 7 (11 May 2020)

II Dirac field

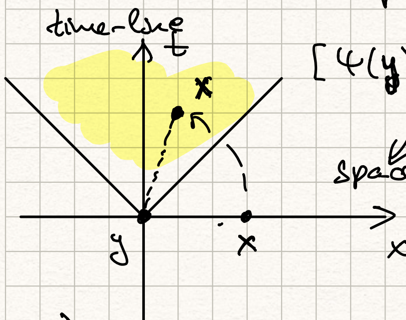
$$\psi(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec p}}} \Big(\underbrace{a(\vec p, s)}_{\text{annih. of par}} u(\vec p, s) e^{-ipx} + \underbrace{b^+(\vec p, s)}_{\text{create antip.}} v(\vec p, s) e^{ipx} \Big)$$

$$\bar\psi(x) = \sum_s \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec p}}} \Big(\underbrace{a^+(\vec p, s)}_{\text{create part}} \bar u(\vec p, s) e^{ipx} + \underbrace{b(\vec p, s)}_{\text{annih. antip.}} \bar v(\vec p, s) e^{-ipx} \Big)$$

$$[a(\vec p, s), a^+(\vec p\,', s')]_+ = (2\pi)^3 \delta(\vec p - \vec p\,') \delta_{ss'}$$

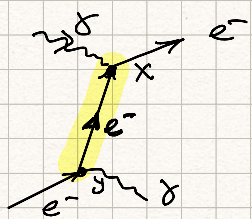
$$[b(\vec p, s), b^+(\vec p\,', s')]_+ = (2\pi)^3 \delta(\vec p - \vec p\,') \delta_{ss'}$$

$$\downarrow$$



$$[\psi(y), \psi^+(x)]_+ = \delta^{(3)}(\vec y - \vec x) \Big|_{y^0 = x^0}$$

space-like



3) Feynman propagator

2-point correlator → Feynman propagator

$$\langle 0 | \psi_\alpha(x) \bar\psi_\beta(y) | 0 \rangle \equiv i S^{(+)}_{\alpha\beta}(x-y)$$

time-like region

$\alpha, \beta = 1, \dots, 4$

$$i S^{(+)}_{\alpha\beta}(x-y) = \sum_s \int \frac{d^3\vec p}{(2\pi)^3\sqrt{2E_{\vec p}}} \sum_{s'} \int \frac{d^3\vec p\,'}{(2\pi)^3\sqrt{2E_{\vec p\,'}}} \langle 0 | \big(a(\vec p, s) u_\alpha(\vec p, s) e^{-ipx} + \cancel{b^+(\vec p, s)} \dots \big) \big(a^+(\vec p\,', s') \bar u_\beta(\vec p\,', s') e^{ip'y} + \cancel{b(\vec p\,', s')} \dots \big) | 0 \rangle$$

$$\big(b(\vec p\,', s') | 0 \rangle = 0 \big)^+$$

$$\langle 0 | b^+(\vec p\,', s') = 0$$

$$\langle 0 | a(\vec p, s) a^+(\vec p\,', s') | 0 \rangle$$

$$+ \langle 0 | a^+(\vec p\,', s') a(\vec p, s) | 0 \rangle = 0$$

$$\parallel$$

$$\underbrace{\langle 0 | [a(\vec p, s), a^+(\vec p\,', s')]_+ | 0 \rangle}_{(2\pi)^3 \delta(\vec p - \vec p\,') \delta_{ss'}}$$

$$i S^{(+)}_{\alpha\beta}(x) = \sum_s \int \frac{d^3\bar p}{(2\pi)^3 (2E_{\bar p})} \, u_\alpha(\bar p, s)\,\bar u_\beta(\bar p, s)\, e^{-ipx} =$$

$y=0$ translat. invar.

$$u_\alpha(\bar p, s)\,\bar u_\beta(\bar p, s) = (\underbrace{\not p}_{\gamma^\mu p_\mu} + m)_{\alpha\beta}$$

$$p^0 = E_{\bar p}$$

$$\partial_\mu e^{-ipx} = -i p_\mu$$

$$= (i\gamma^\mu \partial_\mu + m)_{\alpha\beta} \underbrace{\int \frac{d^3\bar p}{(2\pi)^3 2E_{\bar p}} e^{-ipx}}_{i\Delta(x)}$$

$$S^{(+)}_{\alpha\beta}(x) = (i\gamma^\mu\partial_\mu + m)_{\alpha\beta}\, \Delta(x)$$

$$\langle 0| \bar\psi_\beta(y)\, \psi_\alpha(x) |0\rangle \equiv i S^{(-)}_{\beta\alpha}(x-y)$$

$$i S^{(-)}_{\beta\alpha}(x-y) = \sum_s \int \frac{d^3\bar p}{(2\pi)^3\sqrt{2E_{\bar p}}} \sum_{s'} \int \frac{d^3\bar p'}{(2\pi)^3\sqrt{2E_{\bar p'}}} \Big(\langle 0| (\cancel{a^+ \bar u_\beta e^{ip'y}} + b\,\bar v_\beta e^{-ip'y}) (\cancel{a\, u_\alpha e^{-ipx}} + b^+ v_\alpha e^{ipx}) |0\rangle \Big)$$

$$\langle 0| [b, b^+]_+ |0\rangle = (2\pi)^3 \delta(\bar p - \bar p')\, \delta_{ss'}$$

$$= \sum_s \int \frac{d^3\bar p}{(2\pi)^3 2E_{\bar p}} e^{ip(x-y)} \, \bar v_\beta(\bar p, s)\, v_\alpha(\bar p, s) =$$

$$\bar v_\beta(\bar p, s)\, v_\alpha(\bar p, s) = (\not p - m)_{\alpha\beta}$$

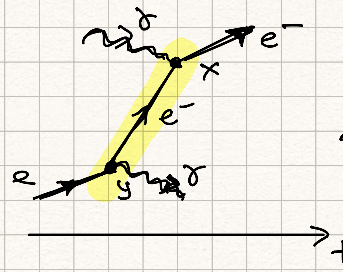
$$= -i(\gamma^\mu\partial_\mu + m)_{\alpha\beta} \cdot \underbrace{\int \frac{d^3\bar p}{(2\pi)^3 2E_{\bar p}} e^{ip(x-y)}}_{\Delta(y-x)}$$

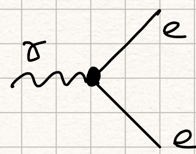
$$S^{(+)}_{\alpha\beta}(x) = (i\gamma^\mu\partial_\mu + m)_{\alpha\beta}\, \Delta(x)$$

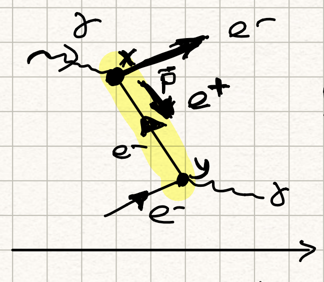
$$S^{(-)}_{\beta\alpha}(x) = -(i\gamma^\mu\partial_\mu + m)_{\alpha\beta}\, \Delta(-x)$$

Feynman propagator

$\gamma e^- \to \gamma e^-$







Propag back in time of particle

$x^0 > y^0$ Propagation of particle from $y \to x$

$y^0 > x^0$ Propag forward in time of antiparticle (e^+)

y: creates e^-
x: annih e^-

x: e^+e^- pair creation
y: e^+ annihilates with initial electron

time product

$$\langle 0 | T(\psi_\alpha(x)\, \bar{\psi}_\beta(y)) | 0 \rangle \equiv i\left(S_F(x-y)\right)_{\alpha\beta}$$

$$T(\psi_\alpha(x)\,\psi_\beta(y)) = \underset{x^0>y^0}{\Theta(x^0-y^0)}\, \psi_\alpha(x)\, \bar{\psi}_\beta(y)$$

$$- \underset{y^0>x^0}{\Theta(y^0-x^0)}\, \bar{\psi}_\beta(y)\, \psi_\alpha(x)$$

$y^0 = 0$ transl. invar. $y=0$

$$\langle 0 | T(\psi_\alpha(x)\, \bar{\psi}_\beta(0)) | 0 \rangle = \Theta(x^0)\, (i\gamma^\mu \partial_\mu + m)_{\alpha\beta}\, \Delta(x)$$
$$+ \Theta(-x^0)\, (i\gamma^\mu \partial_\mu + m)_{\alpha\beta}\, \Delta(-x)$$

$$= i\, \underbrace{(i\gamma^\mu \partial_\mu + m)_{\alpha\beta} \Big(\underbrace{\Theta(x^0)\Delta(x) + \Theta(-x^0)\Delta(-x)}_{\Delta_F(x)} \Big)}_{= S_F(x)_{\alpha\beta}}$$

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \tilde{S}_F(p)$$

$$S_F(x) = (i\gamma^\mu \partial_\mu + m) \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \frac{1}{p^2 - m^2 + i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx}\, \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

$$\tilde{S}_F(p) = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$$

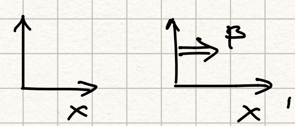
4) Symmetries of the Dirac theory

Lorentz transformation

$$x^\mu \to x'^\mu = a^\mu{}_\nu x^\nu \qquad \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} & a & \\ & \cdots & \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

- Lorentz boost along x-axis

$$a^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$



$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

active boost: change the object

passive boost: change the coord.

$\beta \to -\beta$

- Rotation around z-axis over angle φ

$$a^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$g_{\lambda\mu}$ | $a^\mu{}_\nu$ vs $a_\nu{}^\mu$

$a_{\lambda\nu} \neq a_{\nu\lambda}$ in general

- space inversion

$$(t, \vec{x}) \to (t, -\vec{x})$$

$$a^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

- time reversal

$$(t, \vec{x}) \to (-t, \vec{x})$$

$$a^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \to g^\mu{}_\nu = g^{\mu\sigma} g_{\sigma\nu} = \begin{pmatrix} 1 & & 0 & \\ & 1 & & \\ 0 & & 1 & \\ & & & 1 \end{pmatrix}$$

space-time interval remains invariant

$$x^\mu x_\mu = x'^\mu x'_\mu = \underline{\underline{a^\mu{}_\nu}}\, x^\nu\, \underline{\underline{a_\mu{}^\lambda}}\, x_\lambda$$

$$a^\mu{}_\nu\, a_\mu{}^\lambda = g^\lambda_\nu \quad \rightarrow \quad \det(a^\mu{}_\nu) = \pm 1$$

$(a^2 = 1 \Rightarrow a = \pm 1)$

$\det(a^\mu{}_\nu) = +1$ Proper Lorentz transformation

(3 rotations, 3 boosts) → $SO(3,1)$

preserve time

$\det(a^\mu{}_\nu) = -1$ Improper Lorentz transformation

(combination of proper L.t. with space inversion or time reversal)

$$x^\mu \xrightarrow{a} x'^\mu = a^\mu{}_\nu x^\nu \qquad (i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$x' = a\, x$

$x = a^{-1} x'$

Transformation of spinor field

$S(a), \cancel{S(a)}$

linear transf.

(1) $\psi(x) \xrightarrow{a} \psi'(x') = S(a)\,\psi(x)$ direct

(2) $\psi'(x') \xrightarrow{a^{-1}} \psi(x) = S(a^{-1})\,\psi'(x')$ inverse

$S(a)$ → acts in Dirac space

(4×4) matrix

from (1) → $\psi(x) = S^{-1}(a)\,\psi'(ax)$ $\qquad S^{-1}(a) = S(a^{-1})$

$\psi(x) = S(a^{-1})\,\psi'(ax)$

Lorentz covariance of Dirac eq.

$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

$\partial_\mu = \frac{\partial}{\partial x^\mu}$

$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi(x) = 0$ system S

$\left(i\gamma'^\mu \frac{\partial}{\partial x'^\mu} - m\right)\psi'(x') = 0$ system S'

S and S' related by Lorentz transf $x'^\mu = a^\mu{}_\nu x^\nu$

$$\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = 2g^{\mu\nu} \to \gamma'^\mu = \gamma^\mu$$

$$\gamma'^\mu = U\gamma^\mu U^{-1}$$

$$x'^\nu = a^\nu{}_\mu x^\mu$$

$$S(a)\times \left\{ \left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\underbrace{\psi(x)}_{S^{-1}(a)\psi'(x')} = 0\right.$$

$$\frac{\partial}{\partial x^\mu} = \underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{a^\nu{}_\mu}\frac{\partial}{\partial x'^\nu}$$

$$\Big(i \underbrace{S(a)}_{4\times4}\gamma^\mu S^{-1}(a)\underbrace{\frac{\partial}{\partial x^\mu}}_{a^\nu{}_\mu \frac{\partial}{\partial x'^\nu}} - m\Big)\psi'(x') = 0$$

$$\left(i S(a)\gamma^\mu S^{-1}(a)\, a^\nu{}_\mu \frac{\partial}{\partial x'^\nu} - m\right)\psi'(x') = 0$$

$$\left(i\gamma^\nu \frac{\partial}{\partial x'^\nu} - m\right)\psi'(x') = 0$$

$$S^{-1}(a)\,\{\quad S(a)\gamma^\mu S^{-1}(a)\, a^\nu{}_\mu = \gamma^\nu \quad\}\; S(a)$$

$$a^\nu{}_\mu \gamma^\mu = S^{-1}(a)\gamma^\nu S(a)$$

1) Infinitesimal Lorentz transf $a^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$

$$S(a) = \mathbb{1} - \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$$

2) Finite Lorentz transf $\qquad \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$

$\uparrow$ passive

- Transformation of field operators in Hilbert space under proper Lorentz transf

$\hat{\psi}(x) \qquad |\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\, a^\dagger(\vec{p}, s)|0\rangle$

$$|\vec{p}, s\rangle \xrightarrow{a} |\vec{p}', s'\rangle = U(a)|\vec{p}, s\rangle$$

$$\langle \vec{p}' s'| p' s'\rangle = \langle p s|\underbrace{U^\dagger U}_{\text{unitary transf.}}|p s\rangle$$

$$\langle p s| p s\rangle$$