Lecture 6 (6 May 2020)

II Dirac field

2) Quantization

$$\mathcal{L}_{DIRAC} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$$

$$\Big\downarrow \text{ E.L. eq}$$

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0$$

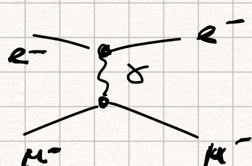
$$\Big\downarrow \; \underline{\text{QFT}}$$

Normal mode expansion

$$\hat{\psi}(x) = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \Big(\underbrace{\hat{a}(\bar{p},s)}_{\substack{\text{annihilate particle}\\ (e^-)\text{ with }(\bar{p},s)}}\, u(\bar{p},s)\, \overbrace{e^{-ipx}}^{\text{positive en. sol}} + \underbrace{\hat{b}^+(\bar{p},s)}_{\substack{\text{creation of antipart.}\\ (e^+)\text{ with }(\bar{p},s)}}\, v(\bar{p},s)\, \overbrace{e^{ipx}}^{\text{neg. en. sol}} \Big)$$

$$\hat{\bar{\psi}}(x) = \sum_s \int \frac{d^3\bar{p}}{(2\pi)^3\sqrt{2E_{\bar{p}}}} \Big(\underbrace{\hat{a}^+(\bar{p},s)}_{\substack{\text{create particle}\\ (e^-)\text{ with }(\bar{p},s)}}\, \bar{u}(\bar{p},s)\, e^{ipx} + \underbrace{\hat{b}(\bar{p},s)}_{\substack{\text{annihilate antipart}\\ (e^+)\text{ with }(\bar{p},s)}}\, \bar{v}(\bar{p},s)\, e^{-ipx} \Big)$$

Single particle (fermion) state



$$|\vec{p}, s\rangle$$

$$|\vec{p}, s\rangle_{e^-} \equiv \sqrt{2E_{\bar{p}}}\; a^+(\bar{p},s)\,|0\rangle$$

$${}_{e^-}\langle \bar{p}, s \,|\, \bar{p}', s'\rangle_{e^-} = (2\pi)^3\, \delta^3(\bar{p}-\bar{p}')\,(2E_{\bar{p}})\,\underline{\delta_{ss'}}$$

$$a(\bar{p},s)\,|0\rangle = 0$$

$$|\vec{p}, s\rangle_{e^+} = \sqrt{2E_{\bar{p}}}\; b^+(\bar{p},s)\,|0\rangle$$

$$b(\bar{p},s)\,|0\rangle = 0$$

$$a^+(\bar p,s)\, a^+(\bar p,s)\,|0\rangle = 0$$
$$b^+(\bar p,s)\, b^+(\bar p,s)\,|0\rangle = 0$$

$$a^+(\bar p,s)\, a^+(\bar p',s')\,|0\rangle = - a^+(\bar p',s)\, a^+(\bar p,s)\,|0\rangle$$

$$\left(a^+(\bar p,s)\, a^+(\bar p',s') + a^+(\bar p',s')\, a^+(\bar p,s)\right)|0\rangle = 0$$

$$[A,B]_+ = AB + BA$$

impose

$$[a^+(\bar p,s), a^+(\bar p',s')]_+ = 0 \qquad [b^+(\bar p,s), b^+(\bar p',s')] = 0$$
$$[a(\bar p,s), a(\bar p',s')]_+ = 0 \qquad \text{the same } b \qquad [b(\bar p,s), b(\bar p',s')] = 0$$

$$[a(\bar p,s), a^+(\bar p',s')]_+ = (2\pi)^3\delta(\bar p-\bar p')\,\delta_{ss'}$$
$$[b(\bar p,s), b^+(\bar p',s')]_+ = (2\pi)^3\delta(\bar p-\bar p')\,\delta_{ss'}$$

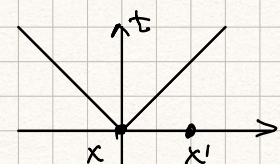
\+ mixed, like e.g. $[a(\bar p,s), b(\bar p',s')] = 0, \ldots$

Equal time anti-com. rel. for the fields

$$[\psi_\alpha(x), \psi^+_\beta(x')]_+\big|_{x^0=x'^0} =$$

$\alpha,\beta = 1,\ldots 4$

(4x4) matrix in Dirac space



$$= \sum_{s,s'} \int \frac{d^3\bar p}{(2\pi)^3\sqrt{2E_{\bar p}}} \int \frac{d^3\bar p'}{(2\pi)^3\sqrt{2E_{\bar p'}}} \Big[a(\bar p,s)\, u_\alpha(\bar p,s)\, e^{-ipx} + b^+(\bar p,s)\, v_\alpha(\bar p,s)\, e^{ipx},$$
$$a^+(\bar p',s')\, u^+_\beta(\bar p',s')\, e^{ip'x'} + b(\bar p',s')\, v^+_\beta(\bar p',s')\, e^{-ip'x'}\Big]_+$$

$$\parallel$$

$$[a(\bar p,s), a^+(\bar p',s')]_+\, u(\bar p,s)\, u^+(\bar p',s')\, e^{-ipx+ip'x'} + [b^+(\bar p,s), b(\bar p',s')]_+\, v(\bar p,s)\, v^+(\bar p',s') \cdot e^{ipx-ip'x'}$$

$$\parallel$$

$$(2\pi)^3\delta(\bar p-\bar p')\,\delta_{ss'} \Big(\underbrace{u(\bar p,s)\, u^+(\bar p,s)}_{\substack{(\not p+m)\gamma^0 \\ (\gamma^0E_{\bar p} - \vec\gamma\vec p + m)\gamma^0}} e^{-ip(x-x')} + \underbrace{v(\bar p,s)\, v^+(\bar p,s)}_{\substack{(\not p-m)\gamma^0 \\ (\gamma^0 E_{\bar p} - \vec\gamma\vec p + m)\gamma^0}} e^{ip(x-x')}\Big) =$$

$$\sum_s u_\alpha(\vec p,s)\,\bar u_\beta(\vec p,s) = (\not p + \overset{\mathbb{1}_{4\times4}}{m})_{\alpha\beta} \quad \Big|\ \gamma^0 \qquad \not p \equiv \gamma^\mu p_\mu = \gamma^0 E_{\vec p} - \vec\gamma\vec p$$

$$\sum_s v(\vec p,s)\,\bar v(\vec p,s) = (\not p - m) \qquad \bar u \equiv u^\dagger\gamma^0$$

$$(\gamma^0)^2 = \mathbb{1}$$

$$\sum_s u(\vec p,s)\,u^\dagger(\vec p,s) = (\not p + m)\gamma^0$$

$$\sum_s v(\vec p,s)\,\bar v(\vec p,s) = (\not p - m)\gamma^0$$

$$-ip(x-x') = -iE_{\vec p}(\overset{0}{\cancel{x^0 - x'^0}}) + i\vec p(\vec x - \vec x')$$

$$\Leftrightarrow \int \frac{d^3\vec p}{(2\pi)^3 2E_{\vec p}} \Big((\gamma^0 E_{\vec p} \cancel{- \vec\gamma\vec p} \cancel{+ m})\gamma^0 e^{i\vec p(\vec x-\vec x')} + \underbrace{(\gamma^0 E_{\vec p} - \vec\gamma\vec p - m)\, e^{-i\vec p(\vec x-\vec x')}}_{\vec p \to -\vec p} \Big)$$

$$E_{\vec p} = \sqrt{\vec p^2 + m^2}$$

$$(\gamma^0 E_{\vec p} \cancel{+ \vec\gamma\vec p} \cancel{- m})\, e^{i\vec p(\vec x-\vec x')}$$

$$[\psi_\alpha(x), \psi^\dagger_\beta(x')]_+ = \int \frac{d^3\vec p}{(2\pi)^3 \cancel{2E_{\vec p}}} \cancel{2E_{\vec p}}\, e^{i\vec p(\vec x-\vec x')} =$$

$$= \delta^{(3)}(\vec x - \vec x')\,\mathbb{1}_{\alpha\beta}$$

Physical quantities

$$H = \int d^3\vec x\ \bar\psi(x)(-i\vec\gamma\vec\nabla + m)\psi(x) =$$

normal mod exp

$$= \int d^3\vec x \sum_{s,s'} \int \frac{d^3\vec p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3\vec p'}{(2\pi)^3\sqrt{2E_{\vec p'}}} \left(a^\dagger(\vec p,s)\bar u(\vec p,s)e^{ipx} + b(\vec p,s)\bar v(\vec p,s)e^{-ipx} \right)$$

$$(-i\vec\gamma\vec\nabla + m)\left(a(\vec p',s')u(\vec p',s')e^{-ip'x} + b^\dagger(\vec p',s')v(\vec p',s')e^{ip'x} \right)$$

$$p'x = E_{\vec p'}t - \vec p'\vec x$$

$$= \int d^3\vec x \sum_{s,s'} \int \frac{d^3\vec p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3\vec p'}{(2\pi)^3\sqrt{2E_{\vec p'}}} \left(a^\dagger(\vec p,s)\bar u(\vec p,s)e^{ipx} + b(\vec p,s)\bar v(\vec p,s)e^{-ipx} \right)$$

$$\Big(a(\vec p',s')\underbrace{(\vec\gamma\vec p' + m)u(\vec p',s')}_{\gamma^0 E_{\vec p'} u(\vec p',s')} e^{-ip'x} + b^\dagger(\vec p',s')\underbrace{(-\vec\gamma\vec p' + m)v(\vec p',s')}_{-\gamma^0 E_{\vec p'} v(\vec p',s')} e^{ip'x} \Big)$$

$$(\not{p}' - m)\, u(\vec p', s') = 0 \qquad (\not{p}' + m)\, v(\vec p', s') = 0$$

$$(\gamma^0 E_{\vec p'} - \underline{\vec\gamma\, \vec p' - m})\, u(\vec p', s') = 0 \qquad (\gamma^0 E_{\vec p'} - \vec\gamma\, \vec p' + m)\, v(\vec p', s') = 0$$

$$(\vec\gamma\, \vec p' + m)\, u(\vec p', s') = \gamma^0 E_{\vec p'}\, u(\vec p', s') \qquad (-\vec\gamma\, \vec p' + m)\, v(\vec p', s') = -\gamma^0 E_{\vec p'}\, v(\vec p', s')$$

$$= \int d^3\vec x \sum_{s,s'} \int \frac{d^3\vec p}{(2\pi)^3\sqrt{2E_{\vec p}}} \int \frac{d^3\vec p'}{(2\pi)^3\sqrt{2E_{\vec p'}}} \Big(e^{i(E_{\vec p} - E_{\vec p'})t - i(\vec p - \vec p')\vec x} \cdot a^\dagger(\vec p, s)\, a(\vec p', s')\, \overbrace{\bar u(\vec p, s)\gamma^0}^{u^\dagger(\vec p, s)} E_{\vec p'}\, u(\vec p', s')$$

$$- e^{i(E_{\vec p} + E_{\vec p'})t - i(\vec p + \vec p')\vec x}\, a^\dagger(\vec p, s)\, b^\dagger(\vec p', s')\, \bar u(\vec p, s)\gamma^0 E_{\vec p'}\, v(\vec p', s')$$

$$+ e^{-i(E_{\vec p} + E_{\vec p'})t + i(\vec p + \vec p')\vec x}\, b(\vec p, s)\, a(\vec p', s')\, \bar v(\vec p, s)\gamma^0 E_{\vec p'}\, u(\vec p', s')$$

$$- e^{-i(E_{\vec p} - E_{\vec p'})t + i(\vec p - \vec p')\vec x}\, b(\vec p, s)\, b^\dagger(\vec p', s')\, \bar v(\vec p, s)\gamma^0 E_{\vec p'}\, v(\vec p', s') \Big)$$

$$\int d^3\vec x\, e^{-i(\vec p - \vec p')\vec x} = (2\pi)^3 \delta^3(\vec p - \vec p')$$

$$= \sum_{ss'} \int \frac{d^3\vec p}{(2\pi)^3 \cancel{2E_{\vec p}}} \cancel{E_{\vec p}} \Big(a^\dagger(\vec p, s)\, a(\vec p, s')\, \overbrace{u^\dagger(\vec p, s)\, u(\vec p, s')}^{\cancel{2}E_{\vec p} \cdot \delta_{ss'}}$$

$$- e^{2iE_{\vec p}t}\, a^\dagger(\vec p, s)\, b^\dagger(-\vec p, s')\, \overbrace{\cancel{u^\dagger(\vec p, s)\, v(-\vec p, s')}}^{0}$$

$$+ e^{2iE_{\vec p}t}\, b(\vec p, s)\, a(-\vec p, s')\, \overbrace{\cancel{v^\dagger(\vec p, s)\, u(-\vec p, s')}}^{0}$$

$$- b(\vec p, s)\, b^\dagger(\vec p', s')\, \underbrace{v^\dagger(\vec p, s)\, v(\vec p, s')}_{\cancel{2}E_{\vec p}\, \delta_{ss'}} \Big) \ominus$$

$$\bar u(\vec p, s)\, u(\vec p, s') = (2m)\, \delta_{ss'}$$

$$u^\dagger(\vec p, s)\, u(\vec p, s') = (2E_{\vec p})\, \delta_{ss'} \qquad u^\dagger(\vec p, s)\, v(-\vec p, s') = 0$$

$$v^\dagger(\vec p, s)\, v(\vec p, s') = (2E_{\vec p})\, \delta_{ss'} \qquad v^\dagger(\vec p, s)\, u(-\vec p, s') = 0$$

$$H = \sum_s \int \frac{d^3\vec p}{(2\pi)^3} E_{\vec p} \Big(a^\dagger(\vec p, s)\, a(\vec p, s) \underline{- \mathbf{b}(\vec p, s)\, \mathbf{b}^\dagger(\vec p, s)} \Big) \ominus$$

$$[b(\vec p, s), b^\dagger(\vec p', s')]_+ = (2\pi)^3 \delta^{(3)}(\vec p - \vec p')\, \delta_{ss'}$$

$$[b(\vec p, s), b^\dagger(\vec p, s)]_+ = (2\pi)^3 \underline{\delta^{(3)}(\mathbf{0})}$$

$$H = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \Big(a^\dagger(\vec{p},s)\, a(\vec{p},s) + b^\dagger(\vec{p},s)\, b(\vec{p},s) - (2\pi)^3 \delta^{(3)}(0) \Big)$$

$$\langle 0 | H | 0 \rangle = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left(-(2\pi)^3 \delta^{(3)}(0) \right)$$

$$H_{phys} = H - \langle 0 | H | 0 \rangle$$

Normal ordering

Physical: define everything relative to a vacuum

mathematics: you introduce a concept of normal order

$$N(a^\dagger a) = a^\dagger a \qquad N(b a b b^\dagger) = - b^\dagger b a b$$

$$N(b^\dagger a) = b^\dagger a$$

$$N(b a^\dagger) = - a^\dagger b$$

odd # of changes "−"
even # of changes "+"

$$N(a a b^\dagger b) = b^\dagger a a b = b^\dagger b a a$$

$$H = \int d^3x\, \bar{\psi}(x)(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi(x)$$

$$\downarrow$$

$$\int d^3x\, N\left(\bar{\psi}(x)(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi(x)\right)$$

$$= \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger(\vec{p},s)\, a(\vec{p},s) + \underline{b^\dagger(\vec{p},s)\, b(\vec{p},s)} \right)$$

$$H - \langle 0 | H | 0 \rangle$$

$$\vec{P} = \int d^3\vec{x}\, N\left(\psi^\dagger(x)(-i\vec{\nabla})\psi(x)\right) = \sum_s \int \frac{d^3 \vec{p}}{(2\pi)^3} \vec{p} \left(a^\dagger a + b^\dagger b \right)$$

Conserved charge

$$\mathcal{L}_{DIRAC} = \bar\psi(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x)$$

symmetry $\Leftrightarrow$ conserved quanti

$$\psi(x) \to \psi'(x) = \underline{e^{i\alpha}}\,\psi(x)$$

$\mathcal{L} \to \mathcal{L}'$ invariant $+ \cancel{\partial_\mu J^\mu}$

$$J^\mu = 0$$

$$e^{i\alpha} = 1 + \underline{i\alpha}$$

$$\underline{\partial_\mu j^\mu = 0}$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\,\delta\psi \;;\quad \delta\psi = i\alpha\psi$$

$$= \bar\psi\, i\gamma^\mu\, i\alpha\psi = (-\alpha)\,\underline{\bar\psi\gamma^\mu\psi}$$

$$\frac{\partial j^0}{\partial t} + \vec\nabla \vec j = 0 \quad \int d^3x \;;\quad Q(t) = \int d^3\vec x\, j^0(\vec x, t) =$$

$$= \int d^3\vec x\left(\underbrace{\bar\psi}_{\psi^\dagger\gamma^0}\gamma^0\psi\right)$$

$$= \int d^3\vec x\, N(\psi^\dagger\psi)$$

normal order oper.

H/W

$$\frac{\partial}{\partial t}\Big(\underbrace{\int d^3x\, j^0}_{Q}\Big) = 0$$

$$Q = \sum_s \int \frac{d^3\vec p}{(2\pi)^3}\left(a^\dagger(\vec p, s)\, a(\vec p, s) - b^\dagger(\vec p, s)\, b(\vec p, s)\right)$$

e^-, e^+

μ^-, μ^+

particles and antiparticles have opposite charges