# Lecture 5 (4 May 2020)

## II Dirac field

MATTER $\nearrow$ Leptons $\begin{pmatrix} e \\ \nu_e \end{pmatrix} \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix} \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix}$

MATTER $\searrow$ quarks $\begin{pmatrix} u \\ d \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix} \begin{pmatrix} t \\ b \end{pmatrix}$

spin $1/2$

### 1) Dirac equation

$$(i\gamma^\mu \partial_\mu - m \overset{\mathbb{1}_{N\times N}}{\downarrow}) \psi(x) = 0 \qquad \psi(x) = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$$

$$E^2 = \vec{p}^{\,2} + m^2$$

$$(\partial_\mu \partial^\mu + m^2)\psi_i(x) = 0$$

$$\Downarrow$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}, \qquad \gamma^\mu = (\gamma^0, \underbrace{\vec{\gamma}}_{\gamma^1, \gamma^2, \gamma^3})$$

$N = 4$ min possible $\qquad \gamma^i$, $i = 1,2,3$

in Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ Pauli (2x2) matrices

Hermiticity condition

$$\left.\begin{aligned} \gamma^{0\dagger} &= \gamma^0 \\ (\gamma^i)^\dagger &= -\gamma^i \\ i &= 1,2,3 \end{aligned}\right| \Rightarrow (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

$$(\gamma^0)^2 = 1$$

$$\gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$$

- $\underline{\gamma_5} \equiv \frac{i}{4!} \underbrace{\varepsilon_{\mu\nu\alpha\beta}}_{\text{total antisym LEVI-CIVITA tensor}} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta$

$\gamma^5; \underline{\underline{\gamma_5}}$

$\varepsilon_{0123} = +1$ ; $\quad \varepsilon^{\mu\nu\alpha\beta} = g^{\mu\mu'} g^{\nu\nu'} g^{\alpha\alpha'} g^{\beta\beta'} \varepsilon_{\mu'\nu'\alpha'\beta'}$

$\varepsilon^{0123} = -1$

$(\gamma_5)^2 = \mathbb{1}_{4\times4}$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$\gamma_5 = \begin{pmatrix} 0 & \mathbb{1}_{2\times2} \\ \mathbb{1}_{2\times2} & 0 \end{pmatrix}$

$\psi(x)$ = complex field

K.G field
$\varphi(x), \varphi^+(x)$

$\psi(x), \quad \bar\psi(x) \equiv \psi^+(x)\gamma^0$

adjoint field

- $\left((i\gamma^\mu\partial_\mu - m)\psi(x) = 0\right)^+$ $\qquad (AB)^+ = B^+A^+$

$-i\,\partial_\mu \psi^+ \underbrace{(\gamma^\mu)^+}_{\gamma^0\gamma^\mu\gamma^0} - m\psi^+ = 0 \quad | \, \gamma^0$

$-i\,\partial_\mu \bar\psi \, \gamma^\mu \cancelto{1}{(\gamma^0)^2} - m\bar\psi = 0$

$i(\partial_\mu\bar\psi)\gamma^\mu + m\bar\psi = 0$ $\qquad$ Dirac eq. for adjoint field

- **Plane-wave solutions**

$\psi^+_{\vec p,s}(x) = e^{-ipx}\, u(\vec p, s)$ $\qquad$ positive energy sol (describe particles)

$\psi^-_{\vec p,s}(x) = e^{ipx}\, v(\vec p, s)$ $\qquad$ negative energy sol. (describe antipart).

$\to \; (\overbrace{\gamma^\mu p_\mu}^{\not p} - m)\, u(\vec p, s) = 0$

$(\gamma^\mu p_\mu + m)\, v(\vec p, s) = 0$

1) Pos. energy sol. $\quad p^\mu = (E_p, \vec{p})$ $\qquad \gamma^\mu p_\mu = \gamma^0 p^0 - \vec{\gamma}\vec{p}$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\chi = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}, \quad \varphi = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$\begin{pmatrix} E_{\vec{p}} - m & -\vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E_{\vec{p}} + m) \end{pmatrix} \begin{pmatrix} \chi_s \\ \varphi_s \end{pmatrix} = 0$$

$$(\vec{\sigma}\vec{p})\chi_s = (E_{\vec{p}} + m)\varphi_s$$

$$\varphi_s = \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}} + m}\chi_s$$

$$u(\vec{p}, s) = N \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}} + m}\chi_s \end{pmatrix}$$

$$s = \pm 1/2, \qquad \chi_s^+ \chi_{s'} = \delta_{ss'}$$

$$\chi_{s_z = +1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \chi_{s_z = -1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\vec{p} \neq 0 \qquad s = \text{helicity}$

$$\left(\frac{\vec{\sigma}}{2}\vec{n}\right)\chi_s = s\,\chi_s, \qquad \vec{n} = \frac{\vec{p}}{|\vec{p}|}$$

- Normalization $\qquad \bar{u} = u^+ \gamma^0$

$$\underbrace{\bar{u}(\vec{p}, s)}_{(1\times 4)} \underbrace{u(\vec{p}, s)}_{(4\times 1)} = N^2 \underbrace{\left(\chi_s^+, \; \chi_s^+\left(\frac{\vec{\sigma}\vec{p}}{E_{\vec{p}} + m}\right)\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\left(\chi_s^+, \; -\chi_s^+\left(\frac{\vec{\sigma}\vec{p}}{E_{\vec{p}} + m}\right)\right)} \cdot \begin{pmatrix} \chi_{s'} \\ \left(\frac{\vec{\sigma}\vec{p}}{E_{\vec{p}} + m}\right)\chi_{s'} \end{pmatrix}$$

$$= N^2 \left( \chi_s^+ \chi_{s'} - \chi_s^+ \frac{\overbrace{(\vec{\sigma}\vec{p})(\vec{\sigma}\vec{p})}^{\vec{p}^2}}{(E_{\vec{p}} + m)^2} \chi_{s'} \right) \ominus$$

$$(\vec{\sigma}\vec{A})(\vec{\sigma}\vec{B}) = \vec{A}\vec{B} + i\vec{\sigma}(\vec{A}\times\vec{B}) \overset{\vec{A}=\vec{B}}{=} \vec{A}^2$$

$\sigma_i A_i \, \sigma_j B_j$

$$\sigma_i \sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k$$

$$\sigma_1 \sigma_2 = i\sigma_3$$

$$E_{\vec{p}}^2 = \vec{p}^2 + m^2$$

$$\vec{p}^2 = E_{\vec{p}}^2 - m^2 = (E_{\vec{p}} - m)(E_{\vec{p}} + m)$$

$$\Leftrightarrow \; N^2 \, \chi_s^+ \chi_{s'} \left( 1 - \frac{E_{\bar p} - m}{E_{\bar p} + m} \right) =$$

$$= N^2 \, \frac{2m}{E_{\bar p} + m} \, \delta_{ss'}$$

- covariant normalization : $N = \sqrt{E_{\bar p} + m}$

$$\bar u(\bar p, s)\, u(\bar p, s) = (2m)\, \delta_{ss'}$$

$$u^+(\bar p, s)\, u(\bar p, s) = (2 E_{\bar p})\, \delta_{ss'}$$

$\Rightarrow$ Lorentz invar phase space

(2) negative energy sol.

$$\psi^-_{\bar p, s}(x) = e^{ipx} \, v(\bar p, s)$$

$$(\gamma_\mu p^\mu + m)\, v(\bar p, s) = 0$$

$$\begin{pmatrix} E_{\bar p} + m & -\vec\sigma \vec p \\ \vec\sigma \vec p & -(E_{\bar p} - m) \end{pmatrix} \begin{pmatrix} \varphi_s' \\ \chi_s' \end{pmatrix} = 0$$

$$(E_{\bar p} + m)\, \varphi_s' = \vec\sigma \vec p \, \chi_s'$$

$$\varphi_s' = \frac{\vec\sigma \vec p}{E_{\bar p} + m} \, \chi_s'$$

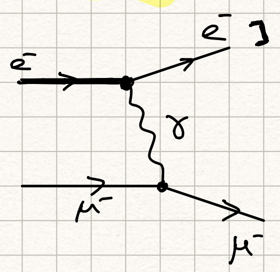
$$v(\bar p, s) = N \begin{pmatrix} \frac{\vec\sigma \vec p}{E_{\bar p} + m} \chi_s' \\ \chi_s' \end{pmatrix}$$

$$\chi'_{s=+1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} ; \quad \chi'_{s=-1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad N = \sqrt{E_{\bar p} + m}$$

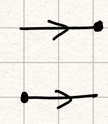
$$\bar v(\bar p, s)\, v(\bar p, s') = -(2m)\, \delta_{ss'}$$

$$v^+(\bar p, s)\, v(\bar p, s') = (2 E_{\bar p})\, \delta_{ss'}$$

• **Spin sums**




$u(\vec p, s)$



$e^{-ipx}\, u(\vec p, s)$

$e^{ipx}\, \bar u(\vec p, s)$

$$u(\vec p, s) = N \begin{pmatrix} \chi_s \\ \frac{\vec\sigma \vec p}{E_p + m}\chi_s \end{pmatrix}$$

$$\sum_{s=\pm 1/2} u(p,s)\,\bar u(\vec p, s) =$$

$$= (E_p + m) \sum_{s=\pm 1/2} \underset{(4\times 1)}{\begin{pmatrix} \chi_s \\ \frac{\vec\sigma \vec p}{E_p + m}\chi_s \end{pmatrix}} \underset{(1\times 4)}{\begin{pmatrix} \chi_s^\dagger & -\chi_s^\dagger \frac{\vec\sigma \vec p}{E_p + m} \end{pmatrix}}$$

$$= (E_p + m) \begin{pmatrix} \sum_s \chi_s \chi_s^\dagger & -\sum_s \chi_s \chi_s^\dagger \frac{\vec\sigma \vec p}{E_p + m} \\ \frac{\vec\sigma \vec p}{E_p + m} \sum_s \chi_s \chi_s^\dagger & -\frac{\vec\sigma \vec p}{E_p + m} \sum_s \chi_s \chi_s^\dagger \frac{\vec\sigma \vec p}{E_p + m} \end{pmatrix} =$$

Completeness relation

$$\sum_s \chi_s \chi_s^\dagger = \mathbb{1}_{2\times 2}$$

$$= (E_p + m) \begin{pmatrix} \mathbb{1} & -\frac{\vec\sigma \vec p}{E_p + m} \\ \frac{\vec\sigma \vec p}{E_p + m} & -\frac{(\vec\sigma \vec p)(\vec\sigma \vec p)}{(E_p + m)^2} \end{pmatrix} =$$

$(\vec\sigma \vec p)(\vec\sigma \vec p) \to \vec p^{\,2} \to E_p^2 - m^2$

$$= \begin{pmatrix} E_p + m & -\vec\sigma \vec p \\ \vec\sigma \vec p & -(E_p - m) \end{pmatrix} = (\gamma^\mu p_\mu + m \mathbb{1}_{4\times 4})$$

$$\gamma^\mu p_\mu = \gamma^0 E_p - \vec\gamma \vec p$$

$p^\mu = (E_p, \vec p) \quad \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ; \vec\gamma = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix}$

$$\sum_{s=\pm 1/2} u(\vec p, s)\,\bar u(\vec p, s) = (\not p + m)$$

$$\sum_{s=\pm 1/2} v(\vec p, s)\,\bar v(\vec p, s) = (\not p - m)$$

## Dirac Lagrangian

$\psi =$ complex field : $\psi, \bar{\psi}$

$$\begin{cases} i\gamma^\mu \partial_\mu \psi - m\psi = 0 \\ i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = 0 \end{cases} \qquad \partial_\mu = (\partial_0, \vec{\nabla})$$

$$\mathcal{L}_{DIRIC} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Euler Lagrange eq.

$$\frac{\partial \mathcal{L}}{\partial \varphi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \psi} = -m\bar{\psi} \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \bar{\psi} i \gamma^\mu \end{cases} \Rightarrow \quad +m\bar{\psi} + i(\partial_\mu \bar{\psi})\gamma^\mu = 0 \quad \checkmark$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\gamma^\mu \partial_\mu - m)\psi \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0 \end{cases} \Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\psi = 0 \quad \checkmark$$

- Conjugate momenta $\qquad \dot{\psi} \equiv \frac{\partial \psi}{\partial t}$

$$\mathcal{L} = \psi^\dagger \gamma^0 \left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\vec{\nabla} - m\right)\psi$$

$$\psi \longrightarrow \pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

$$\bar{\psi} \longrightarrow \bar{\pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

## Hamiltonian

$$\mathcal{H}_{DIRAC} = \pi\dot{\psi} + \cancelto{0}{\dot{\bar{\psi}}\bar{\pi}} - \mathcal{L}$$

$$= \cancel{i\psi^\dagger\dot{\psi}} - \left(\cancel{\psi^\dagger i\dot{\psi}} + \bar{\psi}\, i\vec{\gamma}\vec{\nabla}\psi - m\bar{\psi}\psi\right)$$

$$= -\bar{\psi}(i\vec{\gamma}\vec{\nabla} - m)\psi$$

Energy-mom tensor : $\partial_\mu T^{\mu\nu} = 0 \leftarrow$

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(\partial^\nu\psi) - g^{\mu\nu}\mathcal{L}$$

$$= \bar{\psi}\, i\gamma^\mu(\partial^\nu\psi) - g^{\mu\nu}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

4- momenta

$$P^\nu = \int d^3\vec{x}\, T^{0\nu} = \int d^3x \left(\overset{i\bar{\psi}\gamma^0}{\Pi}\, \underline{\partial^\nu\psi} - g^{0\nu}\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right)$$

$$P^0 = \int d^3x \left(\Pi\, \partial^0\psi - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi\right) = \int d^3x\, \mathcal{H}$$

$$P^i = \int d^3x \left(i\,\psi^\dagger\vec{\nabla}\psi\right)$$

$i = 1,2,3$