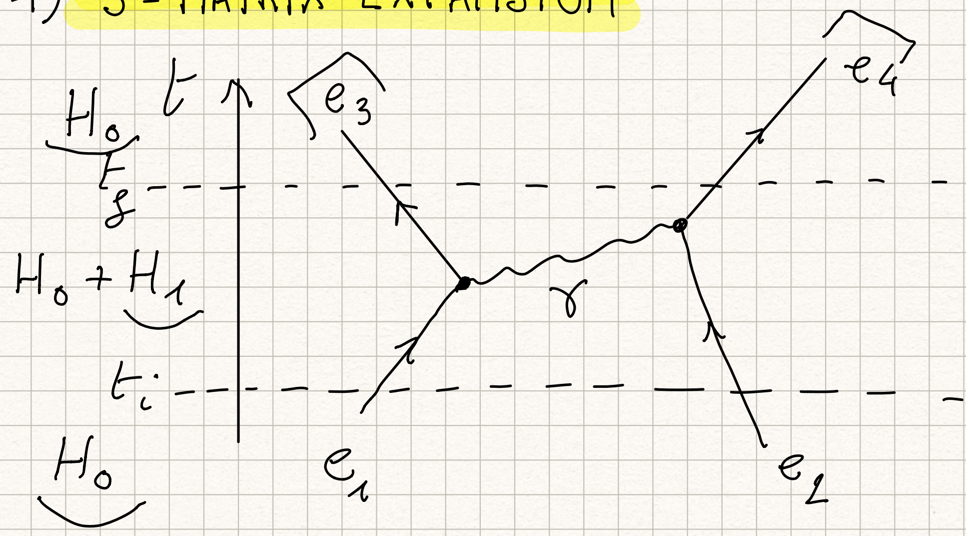
⟹ LECTURE 12

IV INTERACTING FIELDS

1) S-MATRIX EXPANSION



$$H = H_0 + H_1$$

$$|\Psi(t)\rangle_I$$

e.g. $|i\rangle = |e_1^- e_2^-\rangle$

$$|i\rangle = |\Psi(t=-\infty)\rangle_I$$

$$|\Psi(t=+\infty)\rangle_I = S \underbrace{|\Psi(t=-\infty)\rangle_I}_{|i\rangle}$$

$$S^\dagger S = \mathbb{1}$$

$$\langle f \,|\, \Psi(t=+\infty)\rangle_I = \langle f|\, S \,|i\rangle$$

e.g. $|f\rangle = |e_3^- e_4^-\rangle \quad (|e_3^- e_4^- \gamma\rangle)$

$$\underbrace{\langle f|S|i\rangle} = S_{fi}$$

$$\sum_f |S_{fi}|^2 = 1$$

$\hookrightarrow \quad i\frac{d}{dt}|\Psi(t)\rangle_I = \underline{\underline{H_1^I(t)}}\,|\Psi(t)\rangle_I$

$\underset{\sim\sim\sim}{t=+\infty}$

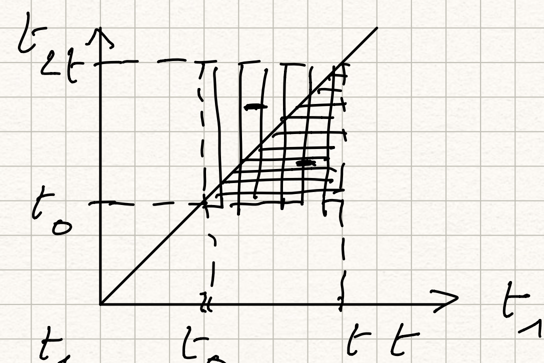
$$S = \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n$$

$$\cdot\ \underbrace{H_1^I(t_1)\cdots\ H_1^I(t_n)} \leftarrow$$

TIME ORDERED $\quad t_1 > t_2 \dots > t_n$

$$\left\{\begin{array}{l} \underline{T\left(H_1^I(t_1)\cdots H_1^I(t_n)\right)} \\ = H_1^I(t_1)\cdots H_1^I(t_n) \\ \text{if } t_1>t_2\dots>t_n \end{array}\right.$$

• $\underline{n=2}$



$$I_2 = \int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{t_1}dt_2\cdots \Rightarrow \int_{t_0}^{t}\underline{\underline{dt_1}}\int_{t_0}^{t_1}dt_2\cdots$$

$t_0\to-\infty,\quad t\to+\infty$

$$T_2 = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T\left(H_1(t_1)\, H_1(t_2) \right)$$

$$t_0 \to -\infty \,, \quad t \to +\infty$$

• n

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \dots \int_{-\infty}^{+\infty} dt_n \; T\left(H_1^I(t_1) \dots H_1^I(t_n) \right)$$

DYSON EXPANSION

$\Rightarrow$ QFT

$$H_1^I(t) = \int d^3\bar{x} \; \mathcal{H}_1^I(\underbrace{t, \bar{x}}_{x})$$

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots \int d^4x_n \; T\left(\mathcal{H}_1(x_1) \dots \mathcal{H}_1(x_n) \right)$$

$\Rightarrow$ ϕ^4 THEORY

$$\mathcal{L}_0 = \mathcal{L}_{KG} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\rightsquigarrow \mathcal{L}_1 = -\frac{\lambda}{4!} \phi^4 \qquad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L}$$

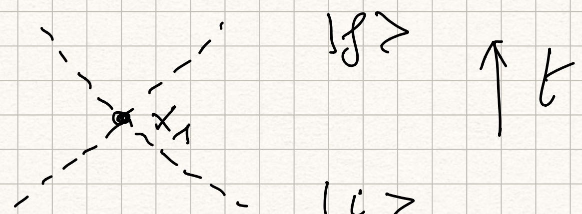
$$\mathcal{H} = \underbrace{\frac{1}{2}\left(\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)}_{\mathcal{H}_0} + \underbrace{\frac{\lambda}{4!}\phi^4}_{\mathcal{H}_1}$$

$$S = \sum_{n=0}^{\infty} \left(-\frac{i\lambda}{4!}\right)^n \frac{1}{n!} \int d^4x_1 \cdots \int d^4x_n \, T\left\{\phi^4(x_1)\cdots\phi^4(x_n)\right\}$$

FOR λ SMALL $\Longrightarrow$ PERTURBATIVE EXPANSION

- 1° ORDER

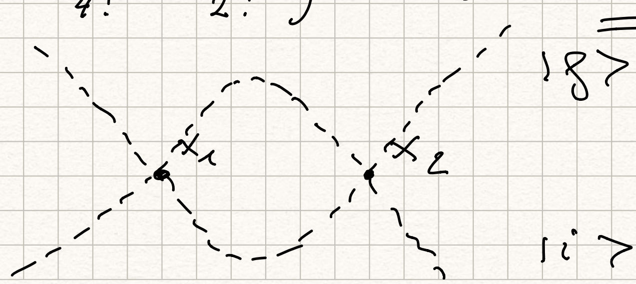
$$S^{(1)} = \left(-\frac{i\lambda}{4!}\right) \int d^4x_1 \; \phi^4(x_1)$$



$\langle f \,|\, S^{(1)} \,|\, i \rangle$ TREE LEVEL

- 2° ORDER

$$S^{(2)} = \left(-\frac{i\lambda}{4!}\right)^2 \frac{1}{2!} \int d^4x_1 \int d^4x_2 \, T\left(\phi^4(x_1)\,\phi^4(x_2)\right)$$



$$\langle f \,|\, S^{(2)} \,|\, i \rangle \rightsquigarrow$$ LOOP CORRECTIONS

QUANTUM "

$\Longrightarrow$ QED (SPIN $1/2$)

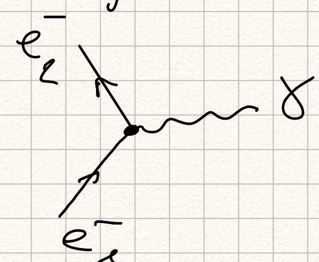
$$\mathcal{L}_0 = \mathcal{L}_{DIRAC} + \mathcal{L}_{MAXWELL}$$

$$\Longrightarrow \mathcal{L}_1 = - \underbrace{e \bar{\psi} \gamma^\mu \psi}_{J^\mu_{em}} A_\mu$$

$$\mathcal{H}_1 = -\mathcal{L}_1 = e N(\bar{\psi} \gamma^\mu \psi A_\mu)$$

$$\langle 0 | \mathcal{H}_1 | 0 \rangle = 0$$

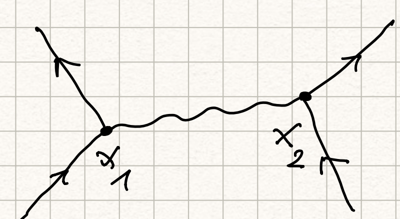
- $$S^{(1)} = -i \int d^4x_1 \; e\, N(\bar{\psi} \gamma^\mu \psi A_\mu)$$



TREE LEVEL $\Longleftarrow$

- $$S^{(2)} = \frac{(-ie)^2}{2!} \int d^4x_1 \int d^4x_2$$

$$\cdot T\left\{ \bar{\psi}(x_1) \gamma^\mu \psi(x_1) A_\mu(x_1) \bar{\psi}(x_2) \gamma^\nu \psi(x_2) A_\nu(x_2) \right\}$$



TREE LEVEL

LOOP DIAGRAM

2) WICK'S THEOREM

$\Rightarrow$ CONTRACTIONS (PROPAGATOR)

QED $\quad \mathcal{H}_1 = \underbrace{e\, N\left(\bar{\psi}(x)\gamma^\mu \psi(x) A_\mu(x)\right)}_{A(x)}$

$$T\left(A(x_1)\, B(x_2)\right)$$

$$= N\left(A(x_1)\, B(x_2)\right) + \underbrace{A(x_1)\, B(x_2)}_{\text{CONTRACTION}}$$

$$\langle 0 | N\left(A(x_1)\, B(x_2)\right) | 0 \rangle = 0$$

$$\langle 0 | T\left(A(x_1)\, B(x_2)\right) | 0 \rangle = \underbrace{A(x_1)\, B(x_2)}$$

- PROPAGATOR

$\rightarrow$ SPIN 0

$$\underbrace{\phi(x_1)\,\phi(x_2)} = \langle 0 | T\left(\phi(x_1)\,\phi(x_2)\right) | 0 \rangle$$

$$= i\, \Delta_F(x_1 - x_2)$$

$\longrightarrow$ SPIN 1/2

$$\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2) = \langle 0|T\left(\psi_\alpha(x_1)\,\bar{\psi}_\beta(x_2)\right)|0\rangle$$

$$= i\left(S_F(x_1-x_2)\right)_{\alpha\beta}$$

$\longrightarrow$ SPIN 1

$$A^\mu(x_1)\,A^\nu(x_2) = \langle 0|T\left(A^\mu(x_1)\,A^\nu(x_2)\right)|0\rangle$$

$$= i\,D_F^{\mu\nu}(x_1-x_2)$$

$\Longrightarrow$ WICK'S THEOREM

$$A_1 = e\bar{\psi}(x_1)\gamma^\mu\psi(x_1)A_\mu(x_1)$$

$$T\{A_1 \dots A_n\}$$

$$= N(A_1 \dots A_n)$$

$$+ N(A_1A_2\dots A_n) + N(A_1A_2A_3\dots A_n)$$

$$+ \dots + N(A_1\dots A_{n-1}A_n)$$

$$+ N(A_1A_2\,A_3A_4\dots A_n) + \dots$$

$$+ N(A_1A_2A_3A_4\dots A_n) + \dots$$

$$+ \dots$$

$\hookrightarrow$ PROOF BY INDUCTION

$\rightsquigarrow$ EXAMPLE ϕ^4

$$T\big(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\big)$$

$$= T(\phi_1\,\phi_2\,\phi_3\,\phi_4)$$

$$= N(\phi_1\,\phi_2\,\phi_3\,\phi_4)$$

$$+ N(\underbrace{\phi_1\,\phi_2}\,\phi_3\,\phi_4) + N(\phi_1\,\phi_2\,\phi_3\,\phi_4)$$

$$+ N(\phi_1\,\phi_2\,\phi_3\,\phi_4) + N(\phi_1\,\underbrace{\phi_2\,\phi_3}\,\phi_4)$$

$$+ N(\phi_1\,\phi_2\,\underbrace{\phi_3\,\phi_4}) \quad + N(\phi_1\,\phi_2\,\phi_3\,\phi_4)$$

$$+ N(\underbrace{\phi_1\,\phi_2}\,\underbrace{\phi_3\,\phi_4}) + N(\phi_1\,\phi_2\,\phi_3\,\phi_4)$$

$$+ N(\phi_1\,\underbrace{\phi_2\,\phi_3}\,\phi_4)$$

$$= N(\phi_1\,\phi_2\,\phi_3\,\phi_4)$$

$$+ i\Delta_F(x_1-x_2)\,N(\phi_3\,\phi_4) + \cdots\, i\Delta_F(x_2-x_4)\,N(\phi_1\phi_3)$$

$$+ \cdots + \big(i\Delta_F(x_1-x_4)\big)\cdot\big(i\Delta_F(x_2-x_3)\big)$$

$$\langle 0|\,T(\phi_1\,\phi_2\,\phi_3\,\phi_4)\,|0\rangle$$

$$= \big(i\Delta_F(x_1-x_4)\big)\big(i\Delta_F(x_2-x_3)\big)$$

$+$ 2 more (FULLY CONTRACTED)