Lecture 10

III The Photon field

2) Quantization of photon field

$$\mathcal{L}_{QED} = \mathcal{L}_{DIRAC} - e(\bar{\psi}\gamma^\mu\psi)A_\mu + \mathcal{L}_{PHOTON}$$

$$\psi(x) \xrightarrow{U(1)} e^{i\theta(x)}\psi(x) \qquad \text{gauge invariant}$$

$$\mathcal{L}_{PHOTON} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \underbrace{\frac{1}{2}(\partial_\mu \hat{A}^\mu)^2}_{\text{gauge fixing term}}$$

$$F_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu$$

- Lorenz gauge $\partial_\mu A^\mu = 0$ (classical level)

Conjugate momenta $\qquad \cancel{\partial_\alpha \hat{A}^\alpha = 0}$

$$\hat{\pi}^\nu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\nu} = -F^{0\nu} - (\partial_\alpha \hat{A}^\alpha)g^{0\nu}$$

$$\hat{\pi}^0 = -\partial_\alpha \hat{A}^\alpha \leftarrow$$

$$\hat{\pi}^i = -F^{0i} = -\partial^0 \hat{A}^i + \partial^i \hat{A}^0$$

ETCR

$$[A^\mu(\bar{x},t), A^\nu(\bar{x}',t)] = 0 \qquad (*)$$

$$[\pi^\mu(\bar{x},t), \pi^\nu(\bar{x}',t)] = 0$$

$$[A^\mu(\bar{x},t), \pi^\nu(\bar{x}',t)] = i g^{\mu\nu}\delta^3(\bar{x}-\bar{x}')$$

$\nu = 0$
$$[A^\mu(\bar{x},t), \pi^0(\bar{x}',t)] = i g^{\mu 0}\delta^3(\bar{x}-\bar{x}')$$

$$[A^\mu(\bar{x},t), \pi^0(\bar{x}',t)] = -[A^\mu(\bar{x},t), \underbrace{\partial_\alpha A^\alpha(\bar{x}',t)}_{\dot{A}^0(\bar{x}',t) + \vec{\nabla}_{x'}\cdot\vec{A}(\bar{x}',t)}] =$$

$$= -[A^\mu(\bar{x},t), \dot{A}(\bar{x}',t)] - \vec{\nabla}_{x'}\cancelto{0}{[A^\mu(\bar{x},t), \vec{A}(\bar{x}',t)]}$$

$\partial_\mu = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$, $A^\mu = (A^0, A^i)$ $\quad \partial^\mu = \frac{\partial}{\partial x_\mu}$

$\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$, $x^\mu = (t, \vec{x})$

$$[A^\mu(\vec{x},t), \dot{A}^0(\vec{x}',t)] = -i g^{\mu 0} \delta^3(\vec{x}-\vec{x}')$$

$\nu = i$

$$[A^\mu(\vec{x},t), \pi^i(\vec{x}',t)] = i g^{\mu i} \delta^3(\vec{x}-\vec{x}')$$

$$[A^\mu(\vec{x},t), \pi^i(\vec{x}',t)] = [A^\mu(\vec{x},t), -\dot{A}^i(\vec{x}',t) + \partial^i A^0(\vec{x}',t)]$$

$$= -[A^\mu(\vec{x},t), \dot{A}^i(\vec{x}',t)] - \nabla^i_{x'} \underbrace{\cancel{[A^\mu(\vec{x},t), A^0(\vec{x}',t)]}}_{0}$$

$$[A^\mu(\vec{x},t), \dot{A}^i(\vec{x}',t)] = -i g^{\mu i} \delta^3(\vec{x}-\vec{x}')$$

$$\rightarrow \quad [A^\mu(\vec{x},t), \dot{A}^\nu(\vec{x},t)] = -i g^{\mu\nu} \delta^3(\vec{x}-\vec{x}')$$

$\mu = \nu = 0$ "$-$"

$\mu = \nu = i$ "$+$"

Normal mode expansion of $A^\mu(\vec{x},t)$

$$A^\mu(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega_k}} \sum_{\lambda=0}^{3} \left(a(\vec{k},\lambda)\, \epsilon^\mu(\vec{k},\lambda)\, e^{-ikx} + a^\dagger(\vec{k},\lambda)\, \epsilon^{\mu *}(\vec{k},\lambda)\, e^{ikx} \right)$$

$k^\mu = \{\omega_k, 0, 0, |\vec{k}|\}$; $\quad \omega_k = |\vec{k}|$, $m_\gamma = 0$

$\vec{k} = |\vec{k}|\, e_z$

classical

$\partial_\mu A^\mu = 0 \rightarrow k_\mu \epsilon^\mu = 0$

linear pol. phot.

$$\epsilon^\mu(\lambda=0) = (1,0,0,0) \quad \text{scalar pol}$$
$$\epsilon^\mu(\lambda=1) = (0,1,0,0)$$
$$\epsilon^\mu(\lambda=2) = (0,0,1,0) \quad \text{transverse pol.}$$
$$\epsilon^\mu(\lambda=3) = (0,0,0,1) \quad \text{long. pol}$$

$\vec{k}\,\vec{\epsilon}(\lambda=1,2) = 0$

$\vec{k} \times \vec{\epsilon}(\lambda=3) = 0$

$$\epsilon^\mu(\vec{k},\lambda)\,\epsilon^*_\mu(\vec{k},\lambda') = -\xi_\lambda \delta_{\lambda\lambda'}$$

$\xi_0 = -1$
$\xi_{1,2,3} = 1$

$a(\vec{k},\lambda)$ annihilates γ with $\vec{k}$ and pol λ
$a^+(\vec{k},\lambda)$ creates γ with $\vec{k}$ and pol λ

pol. vectors can be complex in general (for instance circly pol. photons)

Consider the following combination

$$\int d^3\vec{x}\; e^{-i\vec{k}'\vec{x}} \left(A^\mu(\vec{x},t) + \frac{i}{\omega_{\vec{k}'}} \dot{A}^\mu(\vec{x},t) \right) =$$

$(2\pi)^3\delta(\vec{k}-\vec{k}')$ $\quad$ $(2\pi)^3\delta(\vec{k}+\vec{k}')$

$-i\omega_{\vec{k}}t + i\vec{k}\vec{x}$ $\quad$ $i\omega_{\vec{k}}t - i\vec{k}\vec{x}$

$$= \int d^3\vec{x}\; e^{-i\vec{k}'\vec{x}} \left(\int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=0}^{3} \left(a(\vec{k},\lambda)\epsilon^\mu(\vec{k},\lambda) e^{-ikx} + a^+(\vec{k},\lambda)\epsilon^{*\mu}(\vec{k},\lambda) e^{ikx} \right) \right.$$

$$\left. + \frac{i}{\cancel{\omega_{\vec{k}'}}} \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} (-i\cancel{\omega_{\vec{k}}}) \sum_{\lambda=0}^{3} \left(a(\vec{k},\lambda)\epsilon^\mu(\vec{k},\lambda) e^{-ikx} - a^+(\vec{k},\lambda)\epsilon^{*\mu}(\vec{k},\lambda) e^{ikx} \right) \right)$$

$$= \frac{1}{\sqrt{2\omega_{\vec{k}'}}} \sum_{\lambda=0}^{3} \left(a(\vec{k}',\lambda)\epsilon^\mu(\vec{k}',\lambda) e^{-i\omega_{\vec{k}'}t} + \cancel{a^+(-\vec{k}',\lambda)\epsilon^{\mu*}(-\vec{k}',\lambda) e^{i\omega_{\vec{k}'}t}} \right)$$

$$+ \frac{1}{\sqrt{2\omega_{\vec{k}'}}} \sum_{\lambda=0}^{3} \left(a(\vec{k}',\lambda)\epsilon^\mu(\vec{k}',\lambda) e^{-i\omega_{\vec{k}'}t} - \cancel{a^+(-\vec{k}',\lambda)\epsilon^{\mu*}(-\vec{k}',\lambda) e^{i\omega_{\vec{k}'}t}} \right)$$

$$\left[\begin{array}{l} \int d^3\bar{x}\, e^{-i\bar{k}'\bar{x}} \left(A^\mu(\bar{x},t) + \frac{i}{\omega_{k'}} \dot{A}^\mu(\bar{x},t)\right) = \frac{2}{\sqrt{2\omega_{k'}}} \sum_{\lambda=0}^{3} a(\bar{k}',\lambda)\epsilon^\mu(\bar{k}',\lambda)\, e^{-i\omega_{k'}t} \quad (1) \\ \int d^3\bar{x}\, e^{+i\bar{k}'\bar{x}} \left(A^\mu(\bar{x},t) - \frac{i}{\omega_{k'}} \dot{A}^\mu(\bar{x},t)\right) = \frac{2}{\sqrt{2\omega_{k'}}} \sum_{\lambda=0}^{3} a^+(\bar{k}',\lambda)\epsilon^{\mu *}(\bar{k}',\lambda)\, e^{+i\omega_{k'}t} \quad (2) \end{array}\right.$$

$$\epsilon^\mu(\bar{k},\lambda)\,\epsilon^*_\mu(\bar{k},\lambda') = -\xi_\lambda\, \delta_{\lambda\lambda'}$$

$(1) \cdot \epsilon^*_\mu(\bar{k}',\lambda')$

$(2) \cdot \epsilon_\mu(\bar{k}',\lambda')$

$$\frac{2}{\sqrt{2\omega_{k'}}} \sum_{\lambda=0}^{3} a(\bar{k}',\lambda)\, \delta_{\lambda\lambda'}\, (-\xi_\lambda)\, e^{-i\omega_{k'}t}$$

$$\|$$

$$\frac{2}{\sqrt{2\omega_{k'}}}\, \underline{a(\bar{k}',\lambda')}\, (-\xi_{\lambda'})\, e^{-i\omega_{k'}t}$$

$$a(\bar{k},\lambda) = -\xi_\lambda \frac{\sqrt{2\omega_k}}{2}\, e^{i\omega_k t} \int d^3\bar{x}\, e^{-i\bar{k}\bar{x}}\, \epsilon^*_\mu(\bar{k},\lambda) \left(A^\mu + \frac{i}{\omega_k}\dot{A}^\mu\right)$$

$$a^+(\bar{k},\lambda) = -\xi_\lambda \frac{\sqrt{2\omega_k}}{2}\, e^{-i\omega_k t} \int d^3\bar{x}\, e^{+i\bar{k}\bar{x}}\, \epsilon_\mu(\bar{k},\lambda) \left(A^\mu - \frac{i}{\omega_k}\dot{A}^\mu\right)$$

$$[a(\bar{k},\lambda), a^+(\bar{k}',\lambda')] = \xi_\lambda \xi_{\lambda'} \frac{1}{4}\sqrt{2\omega_k}\sqrt{2\omega_{k'}}\, e^{i(\omega_k-\omega_{k'})t}\, \epsilon^*_\mu(\bar{k},\lambda)\,\epsilon_\nu(\bar{k}',\lambda')$$

$$\int d^3\bar{x} \int d^3\bar{x}'\, e^{-i\bar{k}\bar{x}}\, e^{i\bar{k}'\bar{x}'} \underbrace{\left[A^\mu(\bar{x},t) + \frac{i}{\omega_k}\dot{A}^\mu(\bar{x},t),\, A^\nu(\bar{x}',t) - \frac{i}{\omega_{k'}}\dot{A}^\nu(\bar{x}',t)\right]}_{-\frac{1}{\omega_{k'}} g^{\mu\nu}\delta^{(3)}(\bar{x}-\bar{x}') - \frac{1}{\omega_k} g^{\mu\nu}\delta^{(3)}(\bar{x}-\bar{x}')}$$

$$= \xi_\lambda \xi_{\lambda'} \frac{\sqrt{4\omega_k\omega_{k'}}}{4} (-1)\left(\frac{1}{\omega_{k'}} + \frac{1}{\omega_k}\right) e^{i(\omega_k-\omega_{k'})t} \cdot \epsilon^*_\mu(\bar{k},\lambda)\,\epsilon_\nu(\bar{k}',\lambda') =$$

$$\cdot \underbrace{\int d^3\bar{x}\, e^{-i(\bar{k}-\bar{k}')\vec{x}}}_{(2\pi)^3\delta^3(\bar{k}-\bar{k}')} \cdot g^{\mu\nu}$$

$$= -\xi_\lambda \xi_{\lambda'} \frac{1}{4} 2\omega_k \left(\frac{2}{\omega_k}\right)(2\pi)^3\, \overbrace{\epsilon^*_\mu(\bar{k},\lambda)\epsilon^\mu(\bar{k},\lambda')}^{-\xi_\lambda\delta_{\lambda\lambda'}}$$

$$\xi_\lambda^2 = 1$$

$$\Rightarrow \quad [a(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = \xi_\lambda \, \delta_{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$[a(\vec{k},\lambda), a(\vec{k}',\lambda')] = 0$$

$$[a^\dagger(\vec{k},\lambda), a^\dagger(\vec{k}',\lambda')] = 0$$

$\lambda = 0 \quad \xi_0 = -1$

$\lambda = 1,2,3 \quad \xi_\lambda = +1$ ← Standard Boson com.

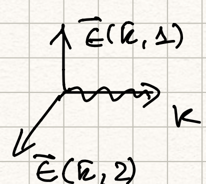
Hamiltonian $\quad H = \int d^3\vec{x}\, \mathcal{H}$

$$\mathcal{H} = \pi_\nu \dot{A}^\nu - \mathcal{L}$$

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3} \omega_k \left(\underline{-a^\dagger(\vec{k},0)\, a(\vec{k},0)} + \underline{\sum_{i=1}^{3} a^\dagger(\vec{k},i)\, a(\vec{k},i)} \right)$$

states with $\lambda = 0$ lead to neg. energy

However,
Physical states: 2 transverse comp. $\lambda = 1,2$



$\lambda = 0,3$ shouldn't apper.

$\cancel{\partial_\mu \hat{A}^\mu = 0}$ ← too strong

$$\partial_\mu \{ \langle 0 | A^\mu(x) A^\nu(y) | 0 \rangle =$$
$$= i D^{\mu\nu}(x-y) =$$
$$= -i g^{\mu\nu} \Delta(x-y)$$

What is the form of Lorenz gauge on the operator level?

$$\langle 0 | \partial_\mu A^\mu(x)\, A^\nu(y) | 0 \rangle =$$
$$= -\underbrace{\partial_\mu g^{\mu\nu} \Delta(x-y)}_{\partial^\nu \quad \neq 0}$$

$$A^\mu(\bar{x},t) = \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=0}^{3} \Big(\underset{(+)}{\underline{a(\bar{k},\lambda)\,\epsilon^\mu(\bar{k},\lambda)\, e^{-ikx}}} + \underset{(-)}{\underline{a^\dagger(\bar{k},\lambda)\,\epsilon^{\mu*}(\bar{k},\lambda)\, e^{ikx}}} \Big)$$

<u>It's enough to impose the following</u>

$|0\rangle$ = standard vacuum $\qquad |\Psi\rangle$ = physical vacuum

def.

$$\underline{\partial_\mu \hat{A}^{\mu(+)}|\Psi\rangle} = 0 \quad \leftarrow \quad \langle\Psi|\,\partial_\mu \hat{A}^{\mu(-)} = 0$$

$$\langle\Psi|\,\partial_\mu \hat{A}^\mu\,|\Psi\rangle = 0$$

$$\partial_\mu \hat{A}^{\mu(+)}|\Psi\rangle = \partial_\mu \int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=0}^{3} a(\bar{k},\lambda)\,\epsilon^\mu(\bar{k},\lambda)\,\underline{e^{-ikx}}\,|\Psi\rangle = 0$$

$$\int \frac{d^3\bar{k}}{(2\pi)^3\sqrt{2\omega_k}} \sum_{\lambda=0}^{3} a(\bar{k},\lambda)\,\underbrace{k_\mu\,\epsilon^\mu(\bar{k},\lambda)}\,e^{-ikx}\,|\Psi\rangle = 0$$

$$\omega_k\,\epsilon^0(\bar{k},\lambda) - |\bar{k}|\,\epsilon^3(\bar{k},\lambda)$$

$$\omega_k\left(\epsilon^0(\bar{k},\lambda) - \epsilon^3(\bar{k},\lambda)\right)$$

$$= \omega_k\left(\delta_{\lambda 0} - \delta_{\lambda 3}\right)$$

$$k^\mu = (\omega_k, 0, 0, |\bar{k}|)$$

$$\epsilon^\mu(\lambda=0) = (1,0,0,0)$$
$$\epsilon^\mu(\lambda=1) = (0,1,0,0)$$
$$\epsilon^\mu(\lambda=2) = (0,0,1,0)$$
$$\epsilon^\mu(\lambda=3) = (0,0,0,1)$$

$$\epsilon^0(\bar{k},\lambda) = \delta_{\lambda 0}$$
$$\epsilon^3(\bar{k},\lambda) = \delta_{\lambda 3}$$

$$\sum_{\lambda=0}^{3} a(\bar{k},\lambda)\left(\delta_{\lambda 0} - \delta_{\lambda 3}\right)|\Psi\rangle = 0$$

$$(a(\vec{k},0) - a(\vec{k},3))|\psi\rangle = 0 \qquad (**)$$

$\uparrow$ fixed

$$\langle\psi|(a^\dagger(\vec{k},0) - a^\dagger(\vec{k},3)) = 0 \qquad (*)$$

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \omega_{\vec{k}} \left(-a^\dagger(\vec{k},0)\,a(\vec{k},0) + \sum_{i=1}^{3} a^\dagger(\vec{k},i)\,a(\vec{k},i) \right)$$

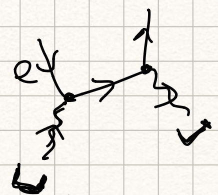
$$\langle\psi|\left(-a^\dagger(\vec{k},0)a(\vec{k},0) + a^\dagger(\vec{k},3)a(\vec{k},3)\right)|\psi\rangle$$

$$= \langle\psi|\big(\underbrace{-a^\dagger(\vec{k},0)a(\vec{k},0) + a^\dagger(\vec{k},3)a(\vec{k},0)} - a^\dagger(\vec{k},3)a(\vec{k},0)$$
$$+ a^\dagger(\vec{k},3)a(\vec{k},3)\big)|\psi\rangle$$

$$= -\langle\psi|(a^\dagger(\vec{k},0) - a^\dagger(\vec{k},3))\,a(\vec{k},0)|\psi\rangle$$
$$- \langle\psi|\, a^\dagger(\vec{k},3)(a(\vec{k},0) - a(\vec{k},3))|\psi\rangle = 0$$

$$\langle\psi|H|\psi\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \omega_{\vec{k}} \sum_{\lambda=1,2} \langle\psi|\, a^\dagger(\vec{k},\lambda)\,a(\vec{k},\lambda)\,|\psi\rangle$$

- Physical quantities only involve transverse photons $\lambda = 1,2$
- Combined energy of scalar & longit. photons is zero



(internal photon)
propagates all states $\Rightarrow |0\rangle$

$\uparrow$ (external photon)
created on top of the physical vacuum $|\Psi\rangle$

3) Feynman Propagator for Photons

Two point correlation function

$$\langle 0 | A^{\mu}(x) A^{\nu}(y) | 0 \rangle \equiv i D^{\mu\nu}(x-y)$$