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# KLEIN-GORDON FIELD

- 1) INTRODUCTION AND MOTIVATION
- 2) ELEMENTS OF CLASSICAL FIELD THEORY
- 3) QUANTIZATION OF KLEIN-GORDON FIELD
- 4) FEYNMAN PROPAGATOR FOR KLEIN-GORDON FIELD

# 1) INTRODUCTION AND MOTIVATION

## ⇒ SPECIAL RELATIVITY

↳ SPACE-TIME POINT DENOTED BY

$$x^\mu (t, \vec{x}) \quad \text{"CONTRAVARIANT"}$$

NOTATION: THROUGHOUT THESE NOTES WE USE NATURAL UNITS  $\hbar = c = 1$

OR  $x_\mu (t, -\vec{x})$  "COVARIANT"

CONNECTED BY

$$x_\mu = g_{\mu\nu} x^\nu \quad \text{OR} \quad x^\mu = g^{\mu\nu} x_\nu$$

FOR MINKOWSKI SPACE-TIME (FLAT SPACE-TIME)  
WE USE METRIC

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

↳ RELATIVISTIC ENERGY-MOMENTUM : 4-MOMENTUM

$$P^\mu (E, \vec{p})$$

$$P^2 \equiv P^\mu P_\mu = E^2 - \vec{p}^2 = m^2 \Leftrightarrow E^2 = \vec{p}^2 + m^2$$

$P^2$  IS LORENTZ INVARIANT

↳ REST MASS OF PARTICLE

⇒ RELATIVISTIC QUANTUM MECHANICS

↳ NON-RELATIVISTIC : SCHRÖDINGER EQ.

$$H \Psi(\vec{x}, t) = i \frac{\partial \Psi}{\partial t} \quad \text{DESCRIBES QUANTUM MECHANICAL PARTICLE}$$

$$H = T + V = -\frac{1}{2m} \nabla^2 + V$$

↳ RELATIVISTIC EQUATION SPIN 0 : KLEIN-GORDON

$$\underline{\underline{(\partial_\mu \partial^\mu + m^2) \Phi(x) = 0}}$$

WITH  $x$  STANDS FOR  $x^\mu$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial t}, \vec{\nabla} \right)$$

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

$$\partial_\mu \partial^\mu \equiv \square = \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (\text{D' ALAMBERTIAN})$$

FOR FREE PARTICLE : SOLUTIONS  $\rightarrow$  PLANE WAVES

$$\Phi(x) = C e^{-i p \cdot x} \quad \text{WITH } p \cdot x = p_\mu x^\mu$$

$$\downarrow$$

$$p^2 = m^2$$

↳ RELATIVISTIC EQUATION SPIN 1/2 : DIRAC  
FOR FREE PARTICLE :

$$\underline{\underline{(i \gamma^\mu \partial_\mu - m) \psi(x) = 0}}$$

WITH  $\gamma^\mu$  : DIRAC MATRICES (4x4 MATRICES)

$$\gamma^\mu = (\gamma^0, \vec{\gamma})$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_{2 \times 2} & 0 \\ 0 & -\mathbb{I}_{2 \times 2} \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

$$\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$$

PAULI MATRICES

AND  $\psi(x)$  : 4-COMPONENT SPINOR

→ POSITIVE ENERGY SOLUTIONS

$$\psi_+(x) = C e^{-i p \cdot x} U(\vec{p}, s) \quad s = \pm \frac{1}{2}$$

$$\hookrightarrow (\not{p} - m) U(\vec{p}, s) = 0$$

$$\Rightarrow U(\vec{p}, s) = \sqrt{E+m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi_s \end{pmatrix}$$

FOR SPIN QUANTIZED ALONG Z-AXIS

$$\chi_{+\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

NORMALIZATION  $\bar{U}(\vec{p}, s) U(\vec{p}, s) = 2m \delta_{ss'}$

→ NEGATIVE ENERGY SOLUTIONS

$$\psi_{-}(x) = C e^{+i p \cdot x} u(\bar{p}, s) \quad \dots \quad s = \pm \frac{1}{2}$$

$$\hookrightarrow (\not{p} + m) u(\bar{p}, s) = 0$$

$$\Rightarrow u(\bar{p}, s) = \sqrt{E+m} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi'_s \\ \chi'_s \end{pmatrix}$$

DIRAC INTERPRETED NEGATIVE - ENERGY SOLUTIONS

AS ANTI - PARTICLES  $e^{-} \leftrightarrow e^{+}$

WHICH WERE THEREAFTER CONFIRMED EXPERIMENTALLY

IF ENOUGH ENERGY IS AVAILABLE  $E > 2m$

$e^{-} e^{+}$  PAIR CAN BE CREATED

↳ LIMITATION OF EQUATION DESCRIBING  
SINGLE PARTICLES

## ⇒ PARTICLE CREATION / ANNIHILATION

↳ e.g. KLEIN-GORDON EQ.

DESCRIBES FREE PARTICLES OF MOMENTUM  $\vec{p}$   
ENERGY  $E = \sqrt{\vec{p}^2 + m^2}$

$$\Phi(x) = \sum_{\vec{p}} \left[ a(\vec{p}) e^{-i\vec{p}\cdot x} + a^*(\vec{p}) e^{+i\vec{p}\cdot x} \right]$$

$\Phi$ : REAL,  $a(\vec{p}) \in \mathbb{C}$  (COMPLEX)

$a(\vec{p})$  DESCRIBES WEIGHT OF MODE  $\vec{p}$   
IN SOLUTION  $\Phi(x)$

↳ SECOND QUANTIZATION

$$\left. \begin{array}{l} a(\vec{p}) \Rightarrow \hat{a}(\vec{p}) \\ a^*(\vec{p}) \Rightarrow \hat{a}^+(\vec{p}) \end{array} \right\} \text{ OPERATORS.}$$

$\hat{a}(\vec{p})$  DESCRIBES ANNIHILATION OF PARTICLE  
WITH MOMENTUM  $\vec{p}$

$\hat{a}^+(\vec{p})$  DESCRIBES CREATION OF PARTICLE  
WITH MOMENTUM  $\vec{p}$

$\hat{a}(\vec{p})$  &  $\hat{a}^+(\vec{p})$  SATISFY COMMUTATION  
RELATIONS  
(SEE FURTHER)

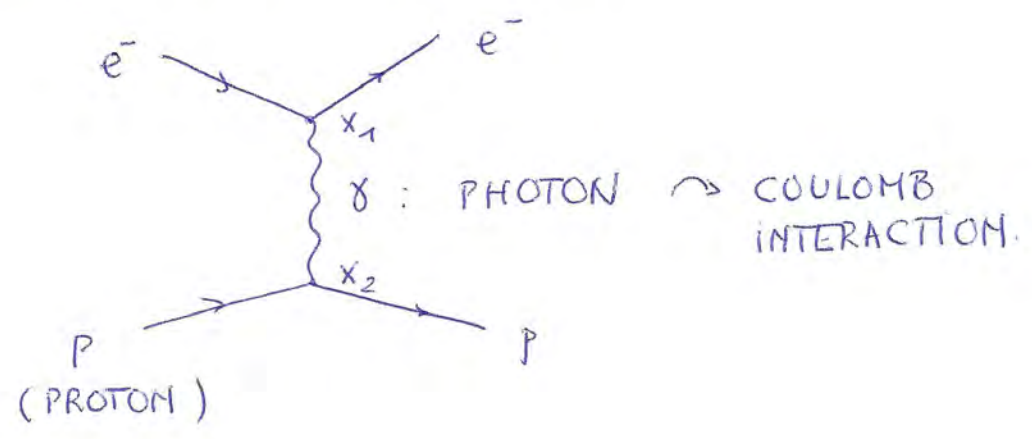
⇒ RELATIVISTIC QUANTUM FIELD THEORY

$\phi(x)$  : WILL THEN ALSO BE INTERPRETED AS FIELD OPERATOR WHICH CAN CREATE / ANNIHILATE PARTICLES IN SPACE-TIME POINT  $x$ .

IMPORTANCE :

↳ DESCRIPTION OF SCATTERING PROCESSES.

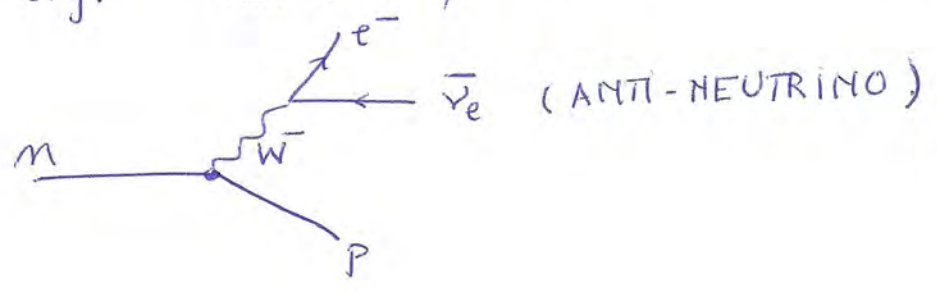
e.g.  $e^- p$  SCATTERING



IN MODERN PHYSICS :

INTERACTIONS = EXCHANGE OF "FORCE-CARRYING" PARTICLES

- 1) ELECTROMAGNETIC : PHOTON (MASSLESS)  $\rightarrow$  LONG RANGE
  - 2) WEAK INTERACTIONS :  $W^+, W^-, Z$  GAUGE BOSONS (MASSIVE)  $\rightarrow$  SHORT RANGE
- e.g. NEUTRON  $\beta$ -DECAY



### 3) STRONG INTERACTIONS

MATTER : SPIN  $\frac{1}{2}$  QUARKS ...

UP  $u$   $e_u = +\frac{2}{3} e$   
 DOWN  $d$   $e_d = -\frac{1}{3} e$   $e$  : PROTON CHARGE

+ 4 HEAVIER  $s, c, b, t$

- PROTON  $P$  CONSISTS OF  $uud$   
 (BUT ALSO  $uud\bar{u}\bar{u}$ ,  $uud\bar{d}\bar{d}$ ,  $uud\bar{u}\bar{u}\bar{u}$ , ...)

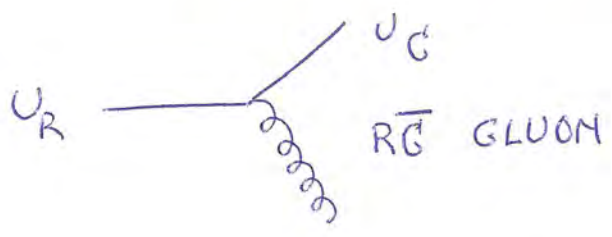
RELATIVISTIC BOUND STATE SYSTEM  
 STILL NOT SOLVED !

- QUARKS COME IN 3 COLORS (RED, GREEN, BLUE)

$u_R, u_G, u_B$

FORCE CARRIERS : SPIN 1 MASSLESS GLUONS

COLOR IS CONSERVED



8 GLUONS  $R\bar{G}, R\bar{B}, G\bar{R}, G\bar{B}, B\bar{R}, B\bar{G},$   
 $R\bar{R} - G\bar{G}, R\bar{R} + G\bar{G} - 2 B\bar{B}$

### 4) GRAVITATIONAL : GRAVITON (SPIN 2) IS EXCHANGED BETWEEN 2 MASSES



↳ DESCRIPTION OF QUANTUM CORRECTIONS

e.g.  $e^-$  ANOMALOUS MAGNETIC MOMENT

$$\vec{\mu} = \frac{e}{2m} \cdot g \cdot \vec{S} \quad \vec{S} = \frac{\vec{\sigma}}{2} \text{ SPIN VECTOR}$$

DIRAC THEORY  $g$ -FACTOR  $g = 2$

QUANTUM FIELD THEORY LEADS TO CORRECTIONS

$$g = 2 \left( 1 + \underset{\substack{\uparrow \\ \text{ANOMALOUS MAGN. MOMENT}}}{\alpha} \right)$$

- 1<sup>st</sup> ORDER CORRECTION TO  $e^-$   $\alpha$



1 LOOP DIAGRAM

$$\alpha^{(1)} = \frac{\alpha}{2\pi} \quad (\text{SCHWINGER}) \quad 1948$$

$$\approx 0.00116$$

$$\approx 1 \cdot 10^{-3}$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

FINE-STRUCTURE CONSTANT

- $\alpha$  IS CALCULATED NOWADAYS UP TO 5<sup>th</sup> ORDER  $\alpha^5$  (5 LOOPS !)

THEORY PRECISION ON  $\alpha$  :  $\sim 5 \cdot 10^{-10}$  !

EXP. " " :  $\sim 6 \cdot 10^{-10}$

## 2) ELEMENTS OF CLASSICAL FIELD THEORY <sup>KG, g</sup>

### ⇒ LAGRANGIAN FIELD THEORY

↳ SYSTEM SPECIFIED BY FIELDS  $\Phi_\tau(x)$

$x$  DENOTES  $x^\mu (t, \vec{x})$

$\tau$  DENOTES COMPONENTS OF FIELD

↳ SCALAR FIELD ( $\tau = 1$ )  $\Phi(x)$

e.g. TEMPERATURE AT GIVEN SPACE-TIME POINT

↳ VECTOR FIELD ( $\tau = 1, 2, 3, 4$ )  $A^\mu(x)$

e.g. ELECTROMAGNETIC 4-VECTOR POTENTIAL

↳ DYNAMICS DESCRIBED BY LAGRANGIAN DENSITY

$$\mathcal{L}(\Phi_\tau, \partial_\mu \Phi_\tau)$$

$$\text{LAGRANGIAN } L \equiv \int d^3\vec{x} \mathcal{L}$$

THROUGH VARIATIONAL PRINCIPLE

$$\text{ACTION } S = \int_{-\Omega} d^4x \mathcal{L}(\Phi_\tau, \partial_\mu \Phi_\tau).$$

CONSIDER VARIATIONS  $\delta\phi_\pi$  OF  $\phi_\pi$

SUCH THAT  $\delta\phi_\pi = 0$  ON BOUNDARY  $\Gamma(\Omega)$

REQUIRE THAT  $S$  IS MINIMIZED  $\delta S = 0$

$\Downarrow$   
FIELD EQUATIONS (EULER-LAGRANGE EQUATIONS)  
FOR  $\phi_\pi$

UNDER VARIATIONS  $\delta\phi_\pi$

$$\delta S = \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\pi} \delta\phi_\pi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\pi)} \delta(\partial_\mu \phi_\pi) \right\}$$

$$= \int_{\Omega} d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_\pi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\pi)} \right) \right\} (\delta\phi_\pi)$$

$$+ \underbrace{\int_{\Omega} d^4x \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\pi)} \delta\phi_\pi \right)}_{\text{BOUNDARY TERM}}$$

BOUNDARY TERM  
VANISHES BECAUSE  $\delta\phi_\pi = 0$  ON  $\Gamma(\Omega)$

$\therefore$  BY REQUIRING THAT

$\delta S = 0$  FOR ARBITRARY  $\delta\phi_\pi$

$$\frac{\partial \mathcal{L}}{\partial \phi_\pi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\pi)} \right) = 0$$

EULER-LAGRANGE  
EQUATIONS



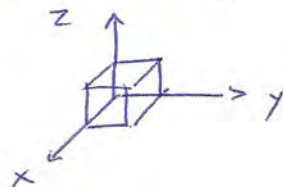
## HAMILTONIAN FIELD THEORY

↳ CONJUGATE MOMENTA :

•  $\phi_{\kappa}(t, \vec{x})$  FIELD

DIVIDE SPACE IN CELLS (DISCRETIZATION)

POSITION OF CELL :  $i = 1, 2, 3, 4, \dots$



GENERALIZED COORDINATES

$$q_{\kappa i}(t) \equiv \phi_{\kappa}(t, i) \quad i = 1, 2, 3, 4, \dots$$

• LAGRANGIAN (DISCRETIZED)

$$L = \int d^3\vec{x} \mathcal{L}$$

$$= \sum_i (\delta\vec{x}_i) \cdot \mathcal{L}(\phi_{\kappa}(t, i), \dot{\phi}_{\kappa}(t, i))$$

↑  
VOLUME ELEMENT

• CONJUGATE MOMENTA

$$P_{\kappa i}(t) = \frac{\partial L}{\partial \dot{q}_{\kappa i}} = \frac{\partial L}{\partial \dot{\phi}_{\kappa}(t, i)}$$

$$\equiv (\delta\vec{x}_i) \cdot \bar{\Pi}_{\kappa}(t, i)$$

$$\bar{\Pi}_{\kappa}(t, i) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{\kappa}(t, i)}$$

↳ HAMILTONIAN

$$H = \sum_{\kappa} \sum_i p_{\kappa i} \dot{q}_{\kappa i} - L$$

$$= \sum_i (\delta \bar{x}_i) \left\{ \sum_{\kappa} \pi_{\kappa}(t, i) \dot{\phi}_{\kappa}(t, i) - \mathcal{L} \right\}$$

↓ CONTINUUM

$$\equiv \int d^3 \bar{x} \quad \mathcal{H}(\phi_{\kappa}(x), \pi_{\kappa}(x))$$

FIELD THEORY : CONTINUUM LIMIT  $\delta \bar{x}_i \rightarrow 0$

$$\boxed{\pi_{\kappa}(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{\kappa}}}$$

$$L(t) = \int d^3 \bar{x} \quad \mathcal{L}(\phi_{\kappa}, \partial_{\mu} \phi_{\kappa})$$

$$H(t) = \int d^3 \bar{x} \quad \mathcal{H}(\phi_{\kappa}, \pi_{\kappa})$$

$$\boxed{\mathcal{H}(x) = \pi_{\kappa}(x) \dot{\phi}_{\kappa}(x) - \mathcal{L}}$$

↳ HAMILTONIAN DENSITY

⇒ EXAMPLE : CLASSICAL KLEIN-GORDON FIELD

$$\hookrightarrow \underline{\underline{\mathcal{L}_{KG} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2}}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi$$

↳ E-L EQUATIONS :  $(\partial_\mu \partial^\mu + m^2) \phi = 0$   
 KLEIN-GORDON EQ.

↳ CONJUGATE MOMENTUM

$$\underline{\underline{\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}}}$$

↳ HAMILTONIAN

$$\underline{\underline{\mathcal{H}_{KG} = \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2}}$$

# ⇒ NOETHER'S THEOREM

## ↳ CONNECTION BETWEEN SYMMETRIES & CONSERVATION LAWS

CONSIDER CONTINUOUS TRANSFORMATION OF FIELDS

$$\Phi_\alpha(x) \rightarrow \Phi'_\alpha(x) = \Phi_\alpha(x) + \delta\Phi_\alpha(x)$$

THIS TF. INDUCES A CHANGE  $\delta\mathcal{L}$  TO  $\mathcal{L}$

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\Phi_\alpha} \delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta(\partial_\mu\Phi_\alpha)$$

↓ E-L EQUATION

$$= \left( \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \right) \delta\Phi_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta(\partial_\mu\Phi_\alpha)$$

$$= \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta\Phi_\alpha \right)$$

THE ABOVE TF. IS SYMMETRY TRANSFORMATION

IF IT LEAVES ACTION INVARIANT  $\delta S = 0$



LAGRANGIAN IS INVARIANT UP TO A 4-DIVERGENCE

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_\mu \mathcal{F}^\mu$$

$$\therefore \partial_\mu \mathcal{F}^\mu = \partial_\mu \left( \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_\alpha)} \delta\Phi_\alpha \right)$$

↓

$$\underline{\underline{\partial_\mu j^\mu = 0}} \quad \text{FOR} \quad \underline{\underline{j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\kappa)} \delta \phi_\kappa - \mathcal{T}^\mu}}$$

↓

CONSERVED CURRENT  $j^\mu$

↓

CONSERVED CHARGE (CONSTANT IN TIME)

$$\underline{\underline{Q \equiv \int d^3 \bar{x} j^0(t, \bar{x})}}$$

$$Q = \int d^3 \bar{x} \left\{ \Pi_\kappa(x) \delta \phi_\kappa(x) - \mathcal{T}^0 \right\}$$

↳ EXAMPLE : SYMMETRY UNDER SPACE-TIME TRANSLATION

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$$

↑  
CONSTANT TRANSLATION

$$\phi'_\kappa(x') = \phi_\kappa(x)$$

IF FIELD AT SHIFTED POSITION = ORIGINAL FIELD AT ORIGINAL POSITION

$$\delta \phi_\kappa(x) = \phi'_\kappa(x) - \phi_\kappa(x)$$

$$= \phi'_\kappa(x) - \phi'_\kappa(x+a)$$

$$= - (\partial_\mu \phi_\kappa) a^\mu$$



$\mathcal{L}$  IS SCALAR  $\Rightarrow$  MUST TRANSFORM IN SAME WAY

$$\underline{\underline{\delta\mathcal{L} = -a^\mu (\partial_\mu \mathcal{L})}}$$

EXPLICIT PROOF:  $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

$$\phi \rightarrow \phi' = \phi - a^\nu (\partial_\nu \phi)$$

$$\partial_\mu \phi \rightarrow \partial_\mu \phi - a^\nu (\partial_\mu \partial_\nu \phi)$$

$$\mathcal{L} \rightarrow \mathcal{L} - a^\nu (\partial_\mu \partial_\nu \phi) (\partial^\mu \phi)$$

$$+ a^\nu m^2 (\partial_\nu \phi) \phi + O(a^2)$$

$$\delta\mathcal{L} \stackrel{!}{=} -a^\nu (\partial_\nu \mathcal{L})$$

$$\therefore \mathcal{J}^\mu = -a^\mu \mathcal{L}$$

$\hookrightarrow$  **CONSERVED CURRENT**

$$j^\mu \equiv -(\partial^\mu \phi_\alpha) a_\nu (\partial^\nu \phi_\alpha) + a_\nu g^{\mu\nu} \mathcal{L}$$

$$\begin{aligned} \partial_\mu j^\mu = 0 &\Leftrightarrow \partial_\mu \left[ (\partial^\mu \phi_\alpha) (\partial^\nu \phi_\alpha) - g^{\mu\nu} \mathcal{L} \right] a_\nu \\ &= 0 \quad \underline{\underline{\forall a_\nu}} \end{aligned}$$



$$\partial_\mu T^{\mu\nu} = 0$$

WITH

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} (\partial^\nu \phi_a) - g^{\mu\nu} \mathcal{L}$$

↳ CONSERVED ENERGY - MOMENTUM TENSOR

$$T^{\mu\nu} = (\partial^\mu \phi_a) (\partial^\nu \phi_a) - g^{\mu\nu} \mathcal{L}$$

↳ CONSERVED CHARGE

4 - MOMENTUM

$$P^\nu = \int d^3\vec{x} T^{0\nu}$$

$$= \int d^3\vec{x} \left\{ \pi_a(x) \partial^\nu \phi_a(x) - g^{0\nu} \mathcal{L} \right\}$$

$$P^0 = \int d^3\vec{x} \left\{ \pi_a(x) \dot{\phi}_a(x) - \mathcal{L} \right\}$$

$$= \int d^3\vec{x} \mathcal{H}(x) = H \quad (\text{ENERGY})$$

$$\vec{P} = - \int d^3\vec{x} \pi_a(x) \vec{\nabla} \phi_a(x) \quad \begin{array}{l} (3\text{-MOMENTUM}) \\ \text{CARRIED BY} \\ \text{FIELD} \end{array}$$

### 3) QUANTIZATION OF KLEIN - GORDON FIELD

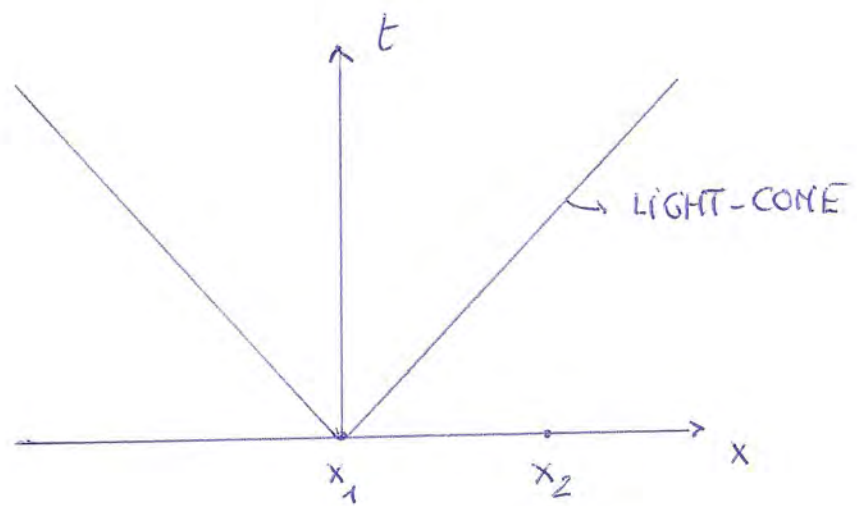
#### ⇒ CANONICAL COMMUTATION RELATIONS

↳ IN QUANTUM THEORY :

WE POSTULATE EQUAL TIME COMMUTATION RELATIONS (ETCR)

FOR FIELDS & CONJUGATE MOMENTA:

$$\begin{aligned} & [\Phi_\alpha(\bar{x}, t), \Phi_\beta(\bar{x}', t)] = 0 \\ & [\pi_\alpha(\bar{x}, t), \pi_\beta(\bar{x}', t)] = 0 \\ & [\Phi_\alpha(\bar{x}, t), \pi_\beta(\bar{x}', t)] = i\delta_{\alpha\beta} \delta^3(\bar{x} - \bar{x}') \end{aligned}$$



FIELDS AT SPACELIKE SEPARATED POINTS  $(x_1 - x_2)^2 < 0$

CANNOT INFLUENCE EACH OTHER (OUTSIDE EACH OTHER'S LIGHT CONE)

↳ COMMUTATORS VANISH

↳ FOR REAL KLEIN-GORDON FIELD

(DESCRIBING NEUTRAL SPIN-0 PARTICLE)

$$[\Phi(\bar{x}, t), \Phi(\bar{x}', t)] = 0$$

$$[\dot{\Phi}(\bar{x}, t), \dot{\Phi}(\bar{x}', t)] = 0$$

$$[\Phi(\bar{x}, t), \dot{\Phi}(\bar{x}', t)] = i \delta^3(\bar{x} - \bar{x}')$$

⇒ NORMAL MODE EXPANSION FOR FIELD  $\Phi$  ( $\Phi = \Phi^\dagger$ )

$$\Phi(\bar{x}, t) = \int \frac{d^3 \bar{k}}{(2\pi)^3} N(\bar{k}) \left\{ a(\bar{k}) e^{-ik \cdot x} + a^\dagger(\bar{k}) e^{+ik \cdot x} \right\}$$

NORMAL MODE  $\bar{k}$  (CONTINUOUS MOMENTUM)

SOLUTIONS OF K-G EQUATION:  $k^2 = m^2$

⇓

$$k^0 = \sqrt{\bar{k}^2 + m^2} \equiv E_{\bar{k}}$$

$a(\bar{k})$  : ANNIHILATES A PARTICLE OF MOM.  $\bar{k}$  & ENERGY  $E_{\bar{k}}$

$a^\dagger(\bar{k})$  : CREATES A PARTICLE OF MOM.  $\bar{k}$  & ENERGY  $E_{\bar{k}}$

$N(\bar{k})$  : CONVENIENTLY CHOSEN NORMALIZATION FACTOR

$$\hookrightarrow i\dot{\Phi}(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} N(\vec{k}) E_{\vec{k}} \left\{ a(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} - a^\dagger(\vec{k}) e^{+i\vec{k}\cdot\vec{x}} \right\}$$

$$\begin{aligned} \hookrightarrow \int d^3\vec{x} e^{-i\vec{k}'\cdot\vec{x}} \Phi(\vec{x}, t) &= \int \frac{d^3\vec{k}}{(2\pi)^3} N(\vec{k}) \left\{ a(\vec{k}) e^{-iE_{\vec{k}}t} \underbrace{\int d^3\vec{x} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{(2\pi)^3 \delta^3(\vec{k}-\vec{k}')} \right. \\ &\quad \left. + a^\dagger(\vec{k}) e^{+iE_{\vec{k}}t} \underbrace{\int d^3\vec{x} e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}}_{(2\pi)^3 \delta^3(\vec{k}+\vec{k}')} \right\} \end{aligned}$$

$$= N(\vec{k}') a(\vec{k}') e^{-iE_{\vec{k}'}t} + N(-\vec{k}') a^\dagger(-\vec{k}') e^{+iE_{-\vec{k}'}t}$$

↓

$$E_{-\vec{k}'} = E_{\vec{k}'}$$

CHOOSE NORMALIZATION

SUCH THAT  $N(-\vec{k}') = N(\vec{k}')$

$$= N(\vec{k}') \left\{ a(\vec{k}') e^{-iE_{\vec{k}'}t} + a^\dagger(-\vec{k}') e^{+iE_{\vec{k}'}t} \right\}$$

↳ ANALOGOUSLY

$$\begin{aligned} \int d^3\vec{x} e^{-i\vec{k}'\cdot\vec{x}} i\dot{\Phi}(\vec{x}, t) &= N(\vec{k}') E_{\vec{k}'} \left\{ a(\vec{k}') e^{-iE_{\vec{k}'}t} - a^\dagger(-\vec{k}') e^{+iE_{\vec{k}'}t} \right\} \end{aligned}$$

$$\hookrightarrow \int d^3 \bar{x} e^{-i \bar{k} \cdot \bar{x}} \left\{ E_{\bar{k}} \phi + i \dot{\phi} \right\}$$

$$= N(\bar{k}) 2E_{\bar{k}} a(\bar{k}) e^{-iE_{\bar{k}} t}$$

$$\int d^3 \bar{x} e^{-i \bar{k} \cdot \bar{x}} \left\{ E_{\bar{k}} \phi - i \dot{\phi} \right\}$$

$$= N(\bar{k}) 2E_{\bar{k}} a^+(-\bar{k}) e^{+iE_{\bar{k}} t}$$

$\Downarrow$  IN 2<sup>o</sup> EQUATION WE CHANGE  $\bar{k} \rightarrow -\bar{k}$

$$a(\bar{k}) = \frac{1}{2E_{\bar{k}} N(\bar{k})} \int d^3 \bar{x} e^{i \bar{k} \cdot \bar{x}} \left\{ E_{\bar{k}} \phi + i \dot{\phi} \right\}$$

WITH  $k^0 = E_{\bar{k}}$

$$a^+(\bar{k}) = \frac{1}{2E_{\bar{k}} N(\bar{k})} \int d^3 \bar{x} e^{-i \bar{k} \cdot \bar{x}} \left\{ E_{\bar{k}} \phi - i \dot{\phi} \right\}$$

WITH  $k^0 = E_{\bar{k}}$

## ⇒ COMMUTATION RELATIONS FOR $a, a^\dagger$

$$\hookrightarrow [a(\vec{k}), a^\dagger(\vec{k}')] = a(\vec{k})a^\dagger(\vec{k}') - a^\dagger(\vec{k}')a(\vec{k})$$

$$= \frac{1}{2E_{\vec{k}} N(\vec{k})} \frac{1}{2E_{\vec{k}'} N(\vec{k}')} \int d^3\vec{x} \int d^3\vec{x}' e^{i\vec{k}\cdot\vec{x}} e^{-i\vec{k}'\cdot\vec{x}'} \quad \text{WITH } \underline{x^0 = x'^0}$$

$$\cdot \underbrace{\left[ E_{\vec{k}} \phi(\vec{x}, t) + i\dot{\phi}(\vec{x}, t), \right.}_{\text{APPLY ETCR's}}$$

$$\left. E_{\vec{k}'} \phi(\vec{x}', t) - i\dot{\phi}(\vec{x}', t) \right]$$

ONLY  $[\phi(\vec{x}, t), \dot{\phi}(\vec{x}', t)]$

IS NON-VANISHING

IS NON-VANISHING

$$E_{\vec{k}} \delta^3(\vec{x} - \vec{x}') + E_{\vec{k}'} \delta^3(\vec{x} - \vec{x}')$$

$$= \frac{1}{2E_{\vec{k}} N(\vec{k})} \frac{1}{2E_{\vec{k}'} N(\vec{k}')} \int d^3\vec{x} e^{i(E_{\vec{k}} - E_{\vec{k}'})t} \cdot (E_{\vec{k}} + E_{\vec{k}'}) e^{-i(\vec{k} - \vec{k}')\cdot\vec{x}}$$

$$\cdot \underbrace{(E_{\vec{k}} + E_{\vec{k}'}) e^{-i(\vec{k} - \vec{k}')\cdot\vec{x}}}_{\downarrow}$$

$$(2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = \frac{1}{2E_{\vec{k}}(N(\vec{k}))^2} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

WE CAN CHOOSE (COVARIANT NORMALIZATION CONVENTION)

$$N(\vec{k}) = \frac{1}{\sqrt{2E_{\vec{k}}}}$$

$$[a(\vec{k}), a^\dagger(\vec{k}')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$[a(\vec{k}), a(\vec{k}')] = 0$$

$$[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0$$

ANALOGOUS RELATIONS  
AS FOR HARMONIC OSCILLATOR



⇒ HAMILTONIAN IN TERMS OF  $a, a^+$

$$\hookrightarrow \mathcal{H} = \frac{1}{2} \left\{ \dot{\phi}^2 + (\bar{\nabla}\phi)^2 + m^2 \phi^2 \right\}$$

$$H = \int d^3 \bar{x} \mathcal{H}(\bar{x}, t)$$



SHOW THAT (EXERCISE!)

$$H = \int \frac{d^3 \bar{k}}{(2\pi)^3} E_{\bar{k}} \left( a^+(\bar{k}) a(\bar{k}) + \frac{\int d^3 \phi(0)}{2} \right)$$

$a^+(\bar{k}) a(\bar{k})$  IS NUMBER OPERATOR

# PARTICLES WITH MOMENTUM  $\bar{k}$

⇒ MOMENTUM IN TERMS OF  $a, a^+$

$$\bar{P} = - \int d^3 \bar{x} \dot{\phi} (\bar{\nabla}\phi)$$



SHOW THAT (EXERCISE!)

$$\bar{P} = \int \frac{d^3 \bar{k}}{(2\pi)^3} \bar{k} a^+(\bar{k}) a(\bar{k})$$

# ⇒ SINGLE PARTICLE STATES

↳ VACUUM  $|0\rangle$  NO PARTICLES.  
 $a(\vec{k})|0\rangle = 0$   
 $\langle 0|0\rangle = 1$  NORMALIZATION

↳ SINGLE PARTICLE STATE

$$|\vec{k}\rangle \equiv \sqrt{2E_{\vec{k}}} a^{\dagger}(\vec{k}) |0\rangle$$

$$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$$

CREATES ONE PARTICLE  
 OF MOMENTUM  $\vec{k}$   
 IN VACUUM

NORMALIZATION CHOSEN SUCH THAT

$$\begin{aligned} \langle \vec{k}' | \vec{k} \rangle &= \sqrt{2E_{\vec{k}'}} \sqrt{2E_{\vec{k}}} \langle 0 | a(\vec{k}') a^{\dagger}(\vec{k}) | 0 \rangle \\ &= \sqrt{2E_{\vec{k}'}} \sqrt{2E_{\vec{k}}} \langle 0 | [a(\vec{k}'), a^{\dagger}(\vec{k})] | 0 \rangle \\ &\text{BECAUSE } a(\vec{k}') | 0 \rangle = 0 \end{aligned}$$

$$\underline{\underline{\langle \vec{k}' | \vec{k} \rangle = (2\pi)^3 2E_{\vec{k}} \delta^3(\vec{k} - \vec{k}')}}$$

$$\hookrightarrow \Phi(\bar{x}, t) | 0 \rangle$$

$$= \int \frac{d^3 \bar{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\bar{k}}}} \left\{ \cancel{a(\bar{k})} e^{-ik \cdot x} + a^\dagger(\bar{k}) e^{+ik \cdot x} \right\} | 0 \rangle$$

0

$$= \int \frac{d^3 \bar{k}}{(2\pi)^3} \frac{e^{ik \cdot x}}{2E_{\bar{k}}} | \bar{k} \rangle$$

CREATES ONE-PARTICLE STATE AT POSITION  $\bar{x}$   
AND TIME  $t$  FROM VACUUM

$\hookrightarrow$  WAVE FUNCTION INTERPRETATION AS IN QM  
 $\langle \bar{x} | \bar{k} \rangle \sim e^{i\bar{k} \cdot \bar{x}}$

$$\begin{aligned} \therefore \left\| \begin{aligned} \langle \bar{k} | \Phi(\bar{x}, t) | 0 \rangle &= e^{ik \cdot x} \\ \langle 0 | \Phi(\bar{x}, t) | \bar{k} \rangle &= e^{-ik \cdot x} \end{aligned} \right. \end{aligned}$$

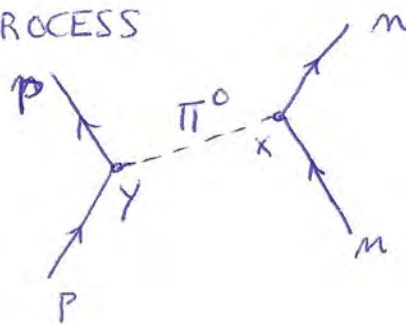
ANNIHILATES A PARTICLE  
AT POSITION  $\bar{x}$  & TIME  $t$   
FROM SINGLE PARTICLE STATE  $|\bar{k}\rangle$

# 4) FEYNMAN PROPAGATOR FOR KLEIN-GORDON FIELD

## ⇒ 2-POINT CORRELATION FUNCTION

↳ AMPLITUDE FOR A PARTICLE TO BE CREATED AT POINT  $y$  AND ANNIHILATED AT POINT  $x$  AT DIFFERENT  $x_0 \neq y_0$

e.g. IN PROCESS



$\pi^0$ : NEUTRAL PION (SPIN 0)

$$\langle 0 | \Phi(x) \Phi(y) | 0 \rangle$$

ONLY  $a^\dagger(\vec{k})$  CONTRIBUTES IN  $\Phi(y)$

↓ "  $a(\vec{k})$  CONTRIBUTES IN  $\Phi(x)$

$$= \langle 0 | \int \frac{d^3 \vec{k}}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} \int \frac{d^3 \vec{k}'}{(2\pi)^3 \sqrt{2E_{\vec{k}'}}} e^{-i\vec{k} \cdot \vec{x} + i\vec{k}' \cdot \vec{y}} a(\vec{k}) a^\dagger(\vec{k}') | 0 \rangle$$

$$\downarrow \langle 0 | a(\vec{k}) a^\dagger(\vec{k}') | 0 \rangle = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{-i\vec{k} \cdot (\vec{x} - \vec{y})} \equiv i\Delta(x - y)$$

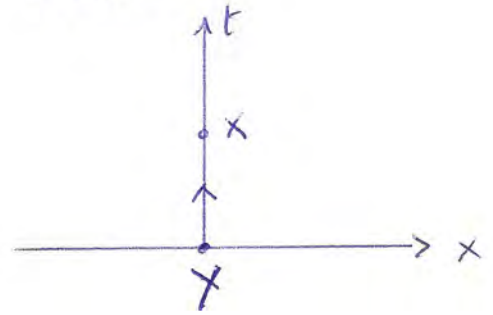
WITH  $E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

↳ NOTE  $\langle 0 | \phi(x) \phi(y) | 0 \rangle \equiv i \Delta(x-y)$

IS ONLY A FUNCTION OF DIFFERENCE  $x-y$   
(TRANSLATIONAL INVARIANCE)

↳ SPECIAL CASES

①  $x^0 - y^0 = t, \bar{x} = \bar{y}$  : PARTICLE AT REST



$$i \Delta(x-y)$$

$$= \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{-iE_{\vec{k}} t}$$

$$= \frac{1}{4\pi^2} \int_0^\infty d|\vec{k}| \frac{|\vec{k}|^2}{E_{\vec{k}}} e^{-iE_{\vec{k}} t}$$

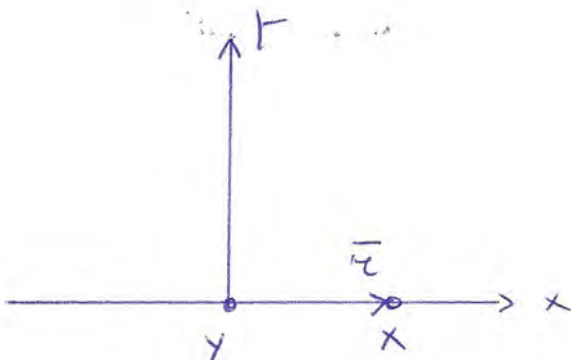
$$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} \qquad \frac{dE_{\vec{k}}}{d|\vec{k}|} = \frac{|\vec{k}|}{E_{\vec{k}}}$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2 - m^2} e^{-iEt}$$

→  $t \rightarrow \infty$   $e^{-imt}$

↳ PARTICLE AT REST WITH MASS  $m$   
PROPAGATING IN TIME

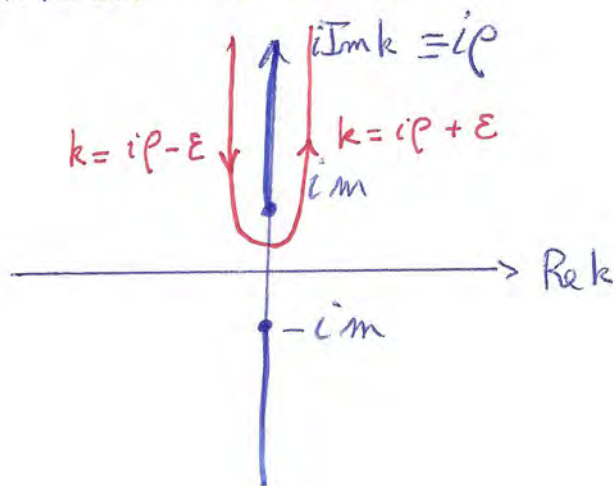
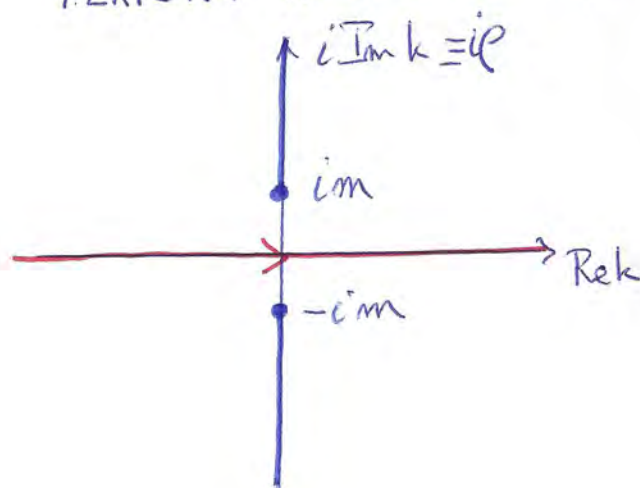
②  $x^0 = y^0, \quad \bar{x} - \bar{y} = \bar{r} : \text{PURE SPATIAL SEPARATION}$

$$\begin{aligned}
 & i \Delta(x-y) \\
 &= \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_k} e^{i\vec{k} \cdot \bar{r}} \\
 & \quad \downarrow k \equiv |\vec{k}| \\
 &= \frac{1}{4\pi^2} \int_0^\infty dk \frac{k^2}{2E_k} \int_{-1}^1 d\cos\theta e^{ikr \cos\theta} \\
 &= \frac{1}{4\pi^2} \frac{1}{i\bar{r}} \int_0^\infty dk \frac{k}{2E_k} \left( e^{ik\bar{r}} - e^{-ik\bar{r}} \right) \\
 &= \frac{1}{8\pi^2 i\bar{r}} \int_{-\infty}^{+\infty} dk \frac{k}{\sqrt{k^2+m^2}} e^{ik\bar{r}}
 \end{aligned}$$


$\frac{1}{\sqrt{k^2+m^2}}$  HAS BRANCH CUTS IN COMPLEX  $k$ -PLANE  
ALONG IM  $k$ -AXIS

FOR  $k > im$  AND  $k < -im$

PERFORM INTEGRAL BY CONTOUR DEFORMATION



$$i \Delta(x-y) \Big|_{x^0=y^0, \bar{x}-\bar{y}=\bar{\tau}}$$

$$= \frac{1}{8\pi^2 i \tau} \left\{ i^2 \int_{-\infty}^m d\rho \frac{\rho e^{-\rho \tau}}{\sqrt{m^2 - \rho^2 - i\epsilon \rho}} + i^2 \int_m^{\infty} d\rho \frac{\rho e^{-\rho \tau}}{\sqrt{m^2 - \rho^2 + i\epsilon \rho}} \right\}$$

$\epsilon$  INFINITESIMAL

$$\sqrt{m^2 - \rho^2 - i\epsilon \rho} = \sqrt{\rho^2 - m^2} \underbrace{e^{-\frac{i\pi}{2}}}_{-i}$$

$$\sqrt{m^2 - \rho^2 + i\epsilon \rho} = \sqrt{\rho^2 - m^2} \underbrace{e^{+\frac{i\pi}{2}}}_{i}$$

$$= \frac{1}{4\pi^2 \tau} \int_m^{\infty} d\rho \frac{\rho e^{-\rho \tau}}{\sqrt{\rho^2 - m^2}} \xrightarrow{\tau \rightarrow \infty} \frac{e^{-m\tau}}{\tau}$$

∞ WE SEE THAT OUTSIDE LIGHT-CONE CORRELATION IS EXPONENTIALLY VANISHING

## ⇒ TIME-ORDERED PRODUCT

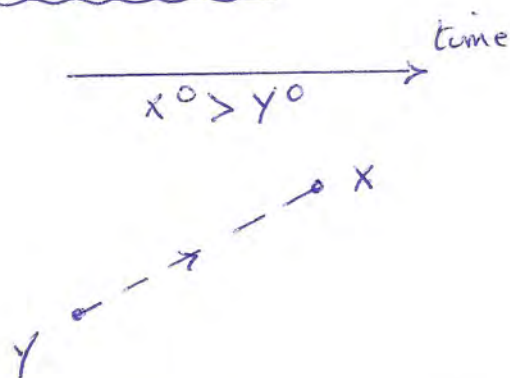
$$T \phi(x) \phi(y) \equiv \theta(x^0 - y^0) \phi(x) \phi(y) + \theta(y^0 - x^0) \phi(y) \phi(x)$$

$$\text{WITH } \theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

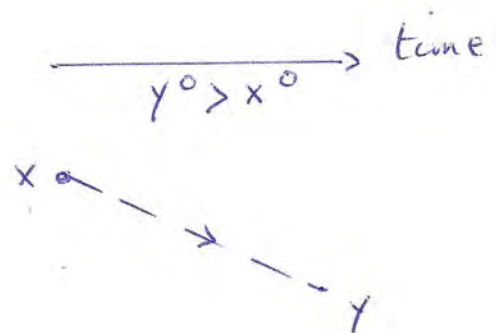
TIME-ORDERED PRODUCT :

FIELD WITH LARGEST TIME ARGUMENT STANDS ON LEFT

SCHEMATICALLY.



INTERPRETATION  
PARTICLE PROPAGATING  
FROM  $y$  TO  $x$   
(FORWARD IN TIME)



PARTICLE PROPAGATING  
FROM  $x$  TO  $y$

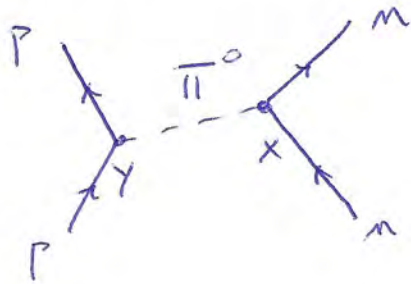


# ⇒ FEYNMAN PROPAGATOR FUNCTION

$$\hookrightarrow \langle 0 | T \phi(x) \phi(y) | 0 \rangle \equiv i \Delta_F(x-y)$$

THIS FUNCTION WILL ENTER PHYSICAL PROCESSES.

→ FEYNMAN DIAGRAMS



$$\begin{aligned} i \Delta_F(x-y) &= \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle \\ &\quad + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle \\ &= \theta(x^0 - y^0) i \Delta(x-y) \\ &\quad + \theta(y^0 - x^0) i \Delta(y-x). \end{aligned}$$

$$i \Delta_F(x-y) = \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} \left\{ \begin{aligned} &\theta(x^0 - y^0) e^{-ik \cdot (x-y)} \\ &+ \theta(y^0 - x^0) e^{+ik \cdot (x-y)} \end{aligned} \right\}$$

WITH  $k^0 = E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

↳ PROPAGATOR IN MOMENTUM SPACE

$$\Delta_F(x) \equiv \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \Delta_F(k^2)$$

WITH  $k^2 = k_\mu k^\mu$

FOURIER TRANSFORM  $\Delta_F(k^2)$  IS GIVEN BY

$$\Delta_F(k^2) = \frac{1}{k^2 - m^2 + i\epsilon}$$

$\epsilon > 0$   
INFINITESIMAL

PROOF: BY CONTOUR DEFORMATION IN COMPLEX  $k^0$  PLANE

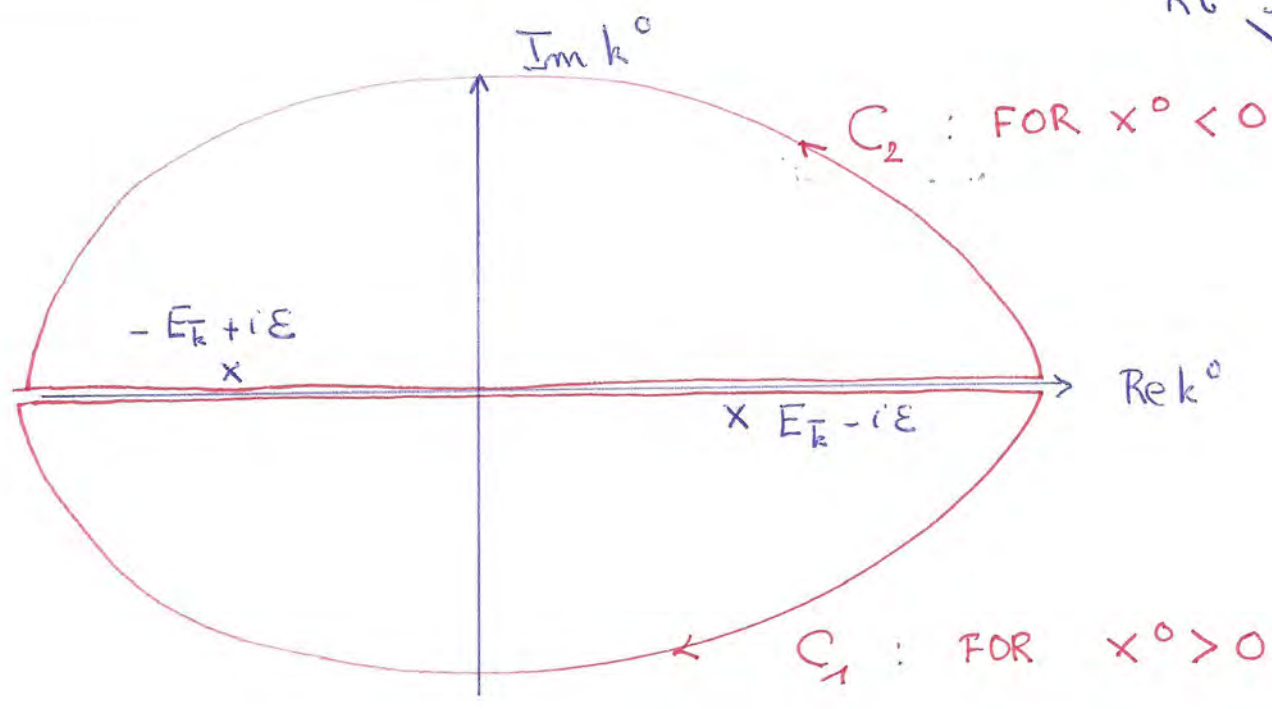
$$\begin{aligned} & k^2 - m^2 + i\epsilon \\ &= (k^0)^2 - (\vec{k}^2 + m^2) + i\epsilon \\ &= (k^0)^2 - E_{\vec{k}}^2 + i\epsilon \end{aligned} \quad \begin{aligned} & \rightarrow E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2} > 0 \\ & \rightarrow \text{FOR } \epsilon \text{ INFINITESIMAL} \end{aligned}$$

$$= (k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)$$

$\Delta_F(k^2)$  HAS 2 POLES IN COMPLEX  $k^0$  PLANE

$$\rightarrow k^0 = E_{\vec{k}} - i\epsilon$$

$$\rightarrow k^0 = -E_{\vec{k}} + i\epsilon$$



• FOR  $x^0 > 0$

CLOSE CONTOUR IN LOWER HALF-PLANE

$e^{-ik^0 x^0}$  REQUIRES  $\text{Im } k^0 < 0$  IN ORDER TO NEGLECT SEMI-CIRCLE AT INFINITY

$$\begin{aligned}
 \underline{x^0 > 0} : \Delta_F(x) &= \int \frac{d^3 \vec{k}}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \\
 &\cdot \int_{-\infty}^{+\infty} dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)} \\
 &= \int \frac{d^3 \vec{k}}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \int_{-\infty}^{+\infty} dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)}
 \end{aligned}$$

$\curvearrowright C_1$

RESIDUE THEOREM ↓

$$-2\pi i \frac{1}{2E_{\vec{k}}} e^{-iE_{\vec{k}} x^0}$$

$$\underline{x^0 > 0} : \Delta_F(x) = -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{-ik \cdot x} \Big|_{k^0 = E_{\vec{k}}}$$

• FOR  $x^0 < 0$

CLOSE CONTOUR IN UPPER HALF-PLANE

$$e^{-ik^0 x^0} \quad \text{Im } k^0 > 0$$

$$x^0 < 0 : \Delta_F(x) = \int \frac{d^3 \vec{k}}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \int dk^0 \frac{e^{-ik^0 x^0}}{(k^0 - E_{\vec{k}} + i\epsilon)(k^0 + E_{\vec{k}} - i\epsilon)}$$

$\underbrace{\hspace{10em}}_{C_2}$   
 $+ 2\pi i \frac{1}{(-2E_{\vec{k}})} \cdot e^{iE_{\vec{k}} x^0}$

$$= -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{iE_{\vec{k}} x^0 + i\vec{k} \cdot \vec{x}}$$

↓ CHANGE INTEGRATION VARIABLE  
 $\vec{k} \rightarrow -\vec{k}$

$$= -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{iE_{\vec{k}} x^0 - i\vec{k} \cdot \vec{x}}$$

$$\underline{x^0 < 0} : \Delta_F(x) = -i \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} e^{ik \cdot x} \Big|_{k^0 = E_{\vec{k}}}$$

• IN TOTAL

$$i \Delta_F(x) = \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_k} \left\{ \begin{array}{l} \theta(x^0) e^{-ik \cdot x} \\ + \theta(-x^0) e^{+ik \cdot x} \end{array} \right\}_{k^0 = E_k}$$

THIS AGREES WITH EXPRESSION OBTAINED ABOVE WHICH COMPLETES PROOF THAT

4D FOURIER TF IS GIVEN BY  $\Delta_F(k^2) = \frac{1}{k^2 - m^2 + i\epsilon}$

↳ FEYNMAN PROPAGATOR AS GREEN'S FUNCTION

$$\Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \frac{1}{k^2 - m^2 + i\epsilon}$$

$$(\square_x + m^2) \Delta_F(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \frac{(-k^2 + m^2)}{k^2 - m^2 + i\epsilon}$$

$$= - \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x}$$

$$= - \delta^4(x)$$

$$(\square_x + m^2) \Delta_F(x) = - \delta^4(x)$$

↓  
GREEN'S FUNCTION OF KG EQUATION

↓  
SOLUTION OF FIELD EQUATION WITH  $-\delta^4(x)$  SOURCE TERM