Theoretical Physics 6a (QFT): SS 2020
Exercise sheet 2

27.04.2020

Exercise 1 (50 points): Real Klein-Gordon field

Using the normal mode expansion of the real Klein-Gordon field

$$\phi(\vec{x},t) \quad = \quad \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left[a(\vec{k})\, e^{-ik\cdot x} + a^\dagger(\vec{k})\, e^{ik\cdot x} \right]$$

with $k^0 = E_{\vec{k}} \equiv \sqrt{\vec{k}^{\,2} + m^2}$, and the equal-time commutation relations

$$\begin{aligned} \left[\phi(\vec{x},t), \phi(\vec{x}',t)\right] &= 0, \\ \left[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}',t)\right] &= 0, \\ \left[\phi(\vec{x},t), \dot{\phi}(\vec{x}',t)\right] &= i\,\delta^{(3)}(\vec{x}-\vec{x}'), \end{aligned}$$

Show that:

(a)(25 points) the Hamiltonian $H = \int d^3\vec{x}\, \frac{1}{2} \left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \right]$ takes the form

$$H = \int \frac{d^3\vec{k}}{(2\pi)^3}\, E_{\vec{k}} \left[a^\dagger(\vec{k}) a(\vec{k}) + \frac{1}{2} \right],$$

(b)(25 points) the momentum $\vec{P} = -\int d^3\vec{x}\, \dot{\phi}\, \vec{\nabla}\phi$ takes the form

$$\vec{P} = \int \frac{d^3\vec{k}}{(2\pi)^3} \vec{k}\, a^\dagger(\vec{k}) a(\vec{k}).$$

Exercise 2 (50 points) : Complex Klein-Gordon field

The complex Klein-Gordon field is used to describe charged bosons. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi, \tag{1}$$

where the field ϕ has the following normal mode expansion

$$\phi(\vec{x}, t) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left[a(\vec{k})\, e^{-ik\cdot x} + b^\dagger(\vec{k})\, e^{ik\cdot x} \right]$$

and satifies the equal-time commutation relations

$$\begin{aligned} \left[\phi(\vec{x}, t), \Pi_\phi(\vec{x}', t)\right] &= i\, \delta^{(3)}(\vec{x} - \vec{x}'), \\ \left[\phi^\dagger(\vec{x}, t), \Pi_{\phi^\dagger}(\vec{x}', t)\right] &= i\, \delta^{(3)}(\vec{x} - \vec{x}'), \end{aligned}$$

all other commutators vanishing. In the following, you can conveniently consider the fields ϕ and $\phi^\dagger$ as independent.

(a)(15 points) Show that (1) is equivalent to the Lagrangian of two independent real scalar fields with same mass and satisfying the standard equal-time commutation relations. *Hint*: Decompose the complex field in real components $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.

(b)(15 points) Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$, and derive the equal-time commutation relations of a, $a^\dagger$, b and $b^\dagger$.

(c)(10 points) Show that (1) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current J^μ.

(d)(10 points) Express the conserved charge $Q = \int d^3x\, J^0$ in terms of creation and annihilation operators. Compute the commutator $[Q, \phi]$.