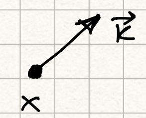
Lecture 4 (29 April 2020)

I Klein Gordon field

$$\hat{\varphi}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3(2E_k)^{1/2}} \left(\underline{\underline{\hat{a}(\vec{k})}}\, e^{-ikx} + \underline{\underline{\hat{a}^+(\vec{k})}}\, e^{ikx} \right)$$

$$\left.\begin{aligned} &[\varphi(t,\vec{x}_1), \varphi(t,\vec{x}_2)] = 0 \\ &[\dot{\varphi}(t,\vec{x}_1), \dot{\varphi}(t,\vec{x}_2)] = 0 \\ &[\varphi(t,\vec{x}_1), \dot{\varphi}(t,\vec{x}_2)] = i\,\delta^{(3)}(x_1 - x_2) \end{aligned}\right| \;\longrightarrow\; \left|\begin{aligned} &[a(\vec{k}), a(\vec{k}')] = 0 \\ &[a^+(\vec{k}), a^+(\vec{k}')] = 0 \\ &[a(\vec{k}), a^+(\vec{k}')] = (2\pi)^3\delta(\vec{k}-\vec{k}') \end{aligned}\right.$$

$$\langle \vec{k} | \varphi(x) | 0 \rangle = \underbrace{e^{ikx}}_{\text{outgoing plane wave}}$$



4) Feynman propagator for KG field

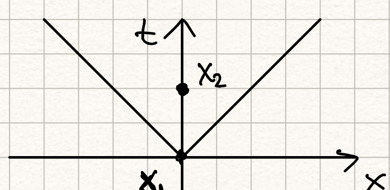
- 2-point correlation function

$$\langle 0 | \varphi(x_2)\varphi(x_1) | 0 \rangle = \int \underbrace{\frac{d^3\vec{k}}{(2\pi)^3 2E_k}} e^{-ik(x_2 - x_1)} \equiv i\Delta(x_2 - x_1)$$

$$\langle 0 | [\varphi(x_2), \varphi(x_1)] | 0 \rangle = i\Delta(x_2 - x_1) - i\Delta(x_1 - x_2)$$

<u>special limits</u>

① time-like region



$\vec{x}_1 = \vec{x}_2$

$0 = x_1^0 < x_2^0 = t$

$$i\Delta(x_2 - x_1) = \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} e^{-iE_k t} \;㊀$$

$E_k = \sqrt{\vec{k}^2 + m^2}$ $\qquad d^3\vec{k} = 4\pi\, |\vec{k}|^2 d|\vec{k}|$

$dE_k = \frac{1}{2E_k} \cdot 2|\vec{k}|\, d|\vec{k}|$

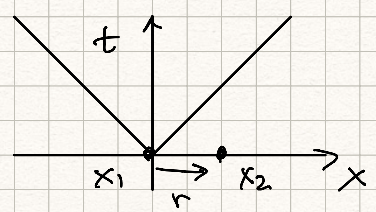
$$㊀\; \frac{1}{4\pi^2} \int_0^\infty d|\vec{k}|\, \frac{|\vec{k}|^2}{E_k} e^{-iE_k t} =$$

$$= \frac{1}{4\pi^2} \int_m^\infty dE_k \sqrt{E_k^2 - m^2}\, e^{-iE_k t} =$$

$|\vec{k}|\, d|\vec{k}| = E_k\, dE_k$

$\overset{t\to\infty}{=} e^{-imt} \ldots$ $\langle 0|[\varphi(x_2)\varphi(x_1)]|0\rangle = (e^{-imt} - e^{imt})\ldots$

② Space-like region



$x_1^0 = x_2^0$

$\vec r_2 = \vec x_2 - \vec x_1 > 0$

$e^{-ik(x_2-x_1)} \to e^{i\vec k\vec r}$

$$i\Delta(\vec r) = \int \frac{d^3\vec k}{(2\pi)^3 2E_k} e^{i\vec k\vec r} =$$

$$d^3\vec k = \overset{k^2}{\overset{\shortparallel}{|\vec k|^2}}\, d\overset{k}{\overset{\shortparallel}{|\vec k|}}\ \overset{d\Omega}{\overbrace{2\pi\, d\cos\theta}}$$

$\vec k\vec r = kr\cos\theta$ $\qquad |\vec r| = r$

$$= \frac{1}{4\pi^2}\int_0^\infty dk\, \frac{k^2}{2E_k}\int_{-1}^{1} d\cos\theta\, e^{ikr\cos\theta} =$$

$E_k = \sqrt{\vec k^2+m^2} = \sqrt{k^2+m^2}$

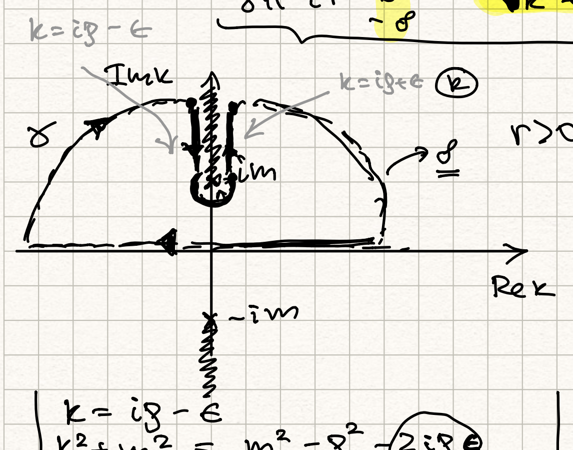
$$= \frac{1}{4\pi^2}\int_0^\infty dk\, \frac{k^2}{2E_k}\cdot\frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$= \frac{1}{8\pi^2 ir}\int_0^\infty dk\, \frac{k}{\sqrt{k^2+m^2}}\left(e^{ikr} - \underline{e^{-ikr}}\right)$$

$$= \underbrace{\frac{1}{8\pi^2 ir}\int_{-\infty}^{+\infty} dk\, \frac{k}{\sqrt{k^2+m^2}}}\, e^{ikr}$$

$e^{i(\mathrm{Re}k + i\mathrm{Im}k)r}$

$e^{-\underline{\mathrm{Im}k}\, r>0}$



$r > 0$

$$\oint_\gamma f(k)\,dk = 0$$

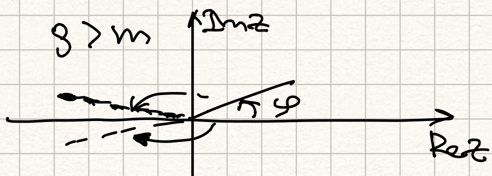
$k = i\varsigma \pm \underline{\epsilon}$, $\epsilon > 0$ $\quad \epsilon \to 0$

$\varsigma > m$

$k = i\varsigma - \epsilon$
$k^2 + m^2 = m^2 - \varsigma^2 - \underset{i\xi}{2i\varsigma\epsilon}$

$k = i\varsigma - \epsilon$
$k^2 + m^2 = m^2 - \varsigma^2 + \underset{i\xi}{2i\varsigma\epsilon}$ $\qquad \xi > 0$

$$i\Delta(\vec r) = \frac{1}{8\pi^2 \cancel{i} r}\left(\cancel{i^2}\int_\infty^m d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{m^2-\rho^2-i\xi}} + i^2 \int_m^\infty d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{m^2-\rho^2+i\xi}}\right)$$



$$z^\alpha = \left(|z|\, e^{i\varphi}\right)^\alpha = |z|^\alpha e^{i\varphi\alpha}$$

$$e^{i\pi}$$

$$\sqrt{m^2-\rho^2-i\xi} = \sqrt{\rho^2-m^2}\; e^{-i\pi\cdot\frac{1}{2}} = -i\sqrt{\rho^2-m^2}$$

$$\sqrt{m^2-\rho^2+i\xi} = \sqrt{\rho^2-m^2}\; e^{i\pi\cdot 1/2} = i\sqrt{\rho^2-m^2}$$

$$i\Delta(\vec r) = \frac{1}{8\pi^2 r}\left\{ -\int_\infty^m d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} + \int_m^\infty d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}}\right)$$

$$= \frac{1}{4\pi^2 r}\int_m^\infty d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \xrightarrow[r\to\infty]{} \text{Const}\,\frac{e^{-mr}}{r}$$

$$\langle 0|[\varphi(x_2)\varphi(x_1)]|0\rangle = \frac{1}{8\pi^2 i r}\left(\int_{-\infty}^{+\infty} \cancel{dk \, \frac{k e^{ikr}}{\sqrt{k^2+m^2}}} - \cancel{\int_{-\infty}^{+\infty} dk \, \frac{k e^{ikr}}{\sqrt{\quad}}} \right)$$

$$= 0$$

Feynman propagator

(time-like region) $\qquad \varphi(x) = \text{real} \quad \pi^0$

Time-ordered product

$$T(\varphi(x)\varphi(y)) = \underbrace{\theta(x^0-y^0)\,\varphi(x)\varphi(y)} + \theta(y^0-x^0)\,\underline{\varphi(y)\varphi(x)}$$

$$\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

$$\langle 0|T(\varphi(x)\varphi(y)|0\rangle \equiv i\Delta_F(x-y)$$

Feynman propagator

$$i\Delta_F(x-y) = \theta(x^0-y^0)\int\frac{d^3\vec{k}}{(2\pi)^3 2E_k}\, e^{-ik(x-y)} + \theta(y^0-x^0)\int\frac{d^3\vec{k}}{(2\pi)^3 2E_k}\, e^{ik(x-y)}$$

$$\Big|\, k^0 = \underbrace{E_k}_{\sqrt{\vec{k}^2+m^2}}$$

$\updownarrow$

Fourier transf.

$$\Delta_F(x-y) = \int\frac{d^4k}{(2\pi)^4}\, e^{-ik(x-y)}\, \Delta_F(k)$$

$k^0, \vec{k}$ running variables

$$\Delta_F(k) = \frac{1}{k^2-m^2+i\varepsilon}, \quad \varepsilon>0$$

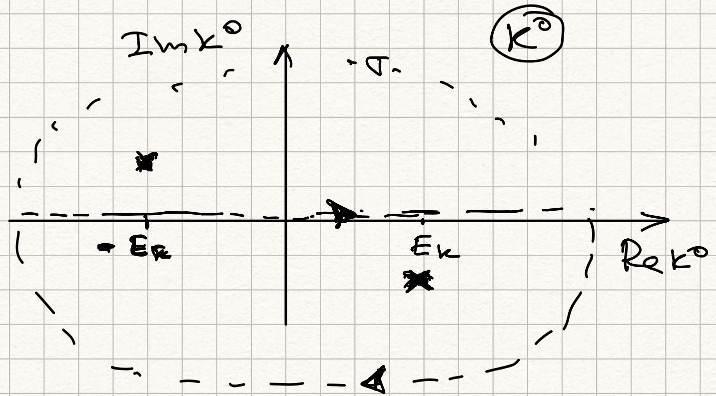
$$\Delta_F(x) = \underbrace{\int\frac{d^4k}{(2\pi)^4}}_{\int_{-\infty}^{+\infty}\frac{dk^0}{(2\pi)}\int\frac{d^3\vec{k}}{(2\pi)^3}}\, e^{-ikx}\cdot\frac{1}{k^2-m^2+i\varepsilon}$$

$$k^2-m^2+i\varepsilon = (k^0)^2 - \overbrace{\vec{k}^2 - m^2}^{E_k^2} + i\varepsilon =$$

$$= (k^0)^2 - E_k^2 + i\varepsilon = 0$$

$$(k^0 - E_k + i\xi)(k^0 + E_k - i\xi) = (k^0)^2 - E_k^2 + \underbrace{2i\xi E_k}_{i\varepsilon}$$

$$k^0 = \pm(E_k - i\xi)$$



$$\Delta_F(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}} \underbrace{\int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} e^{-ik^0x^0} \frac{1}{(k^0-E_k+i\varepsilon)(k^0+E_k-i\varepsilon)}}_{-2\pi i \frac{e^{-iE_k x^0}}{2E_k}}$$

1) $x^0 > 0$

$\Rightarrow$ choose lower plane

contour direction

$$\Delta_F(x) = -i\int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} \underbrace{e^{-iE_k x^0} e^{i\vec{k}\vec{x}}}_{e^{-ikx}} \Big|_{k^0=E_k}$$

2) $x^0 < 0$

$\Rightarrow$ choose upper plane

$$\Delta_F(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\vec{x}} \frac{i}{-2E_k} e^{iE_k x^0}$$

$\vec{k} \to -\vec{k}$

$$= -i\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E_k} e^{-i\vec{k}\vec{x}} e^{iE_k x^0} \qquad E_k = \sqrt{\vec{k}^2+m^2}$$

$$= -i\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E_k} e^{ikx} \Big|_{k^0=E_k}$$

$$i\Delta_F(\underbrace{x}_{x-y}) = \Theta(x^0) \int \frac{d^3k}{(2\pi)^3 2E_k} e^{-ikx} \Big|_{k^0=E_k}$$

$$\Theta(-x^0) \int \frac{d^3k}{(2\pi)^3 2E_k} e^{ikx} \Big|_{k^0=E_k}$$

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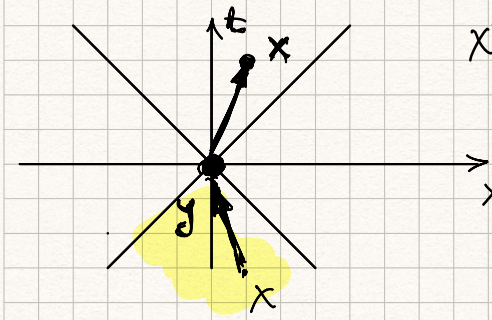
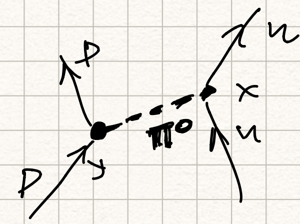
$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2-m^2+i\varepsilon} \qquad \varepsilon > 0 \to 0$$



$x^0 > y^0$



Mathematical meaning: Green's function

$$(\partial_\mu \partial^\mu + m^2)\, \varphi(x) = 0$$

$$(\partial_\mu \partial^\mu + m^2) \cdot \Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \cdot \frac{\cancel{(-k^2+m^2)}^{(-1)}\, e^{-ikx}}{\cancel{k^2-m^2}+i\varepsilon} = -\delta^{(4)}(x)$$

$$\boxed{(\partial_\mu \partial^\mu + m^2)\, \Delta_F(x) = -\delta^{(4)}(x)}$$