Lecture 3 (27 April 2020)

I Klein Gordon field

classical field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

$$\Big\downarrow \text{E.L. eq}$$

$$(\partial_\mu\partial^\mu + m^2)\varphi = 0$$

conjugate momenta

$$\Pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} \quad ; \quad \dot{\varphi} \equiv \frac{\partial\varphi}{\partial t}$$

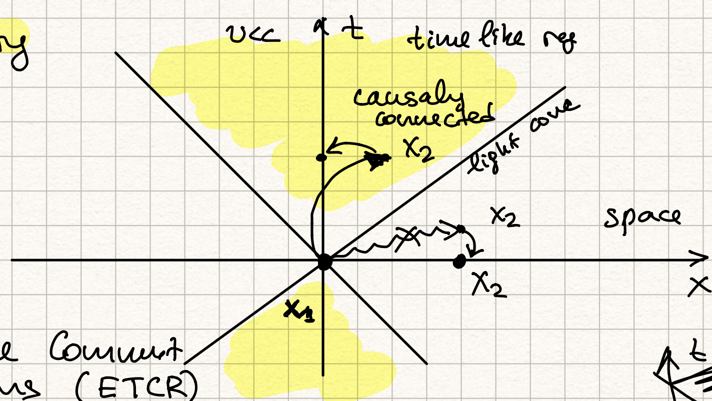
$\varphi(x)$, $\Pi(x)$ numbers $\xrightarrow[\text{QFT}]{}$ operators

3) Quantization of Klein Gordon field

$\hat{\varphi}(\underline{\underline{x}})$ → should encode particle creation / annihilation

rel. QM: $E > 2m$ ~~e^+e^- can be created~~

causality



Equal Time Commut Relations (ETCR)

$$\begin{cases} [\varphi(t,\vec{x}_1), \varphi(t,\vec{x}_2)] = 0 \\ [\Pi(t,\vec{x}_1), \Pi(t,\vec{x}_2)] = 0 \\ [\varphi(t,\underline{\underline{\vec{x}_1}}), \Pi(t,\underline{\underline{\vec{x}_2}})] = i\,\underbrace{\delta^{(3)}(\vec{x}_1-\vec{x}_2)}_{=0 \;\; \vec{x}_1 \neq \vec{x}_2} \end{cases}$$

$[A,B] = AB - BA$

QM $\quad [q_i, p_j] = i\delta_{ij}$

$\downarrow \qquad \downarrow$

$\delta^{(3)}(\vec{x}_1 - \vec{x}_2)$

for. K.G. field : $\Pi = \dot{\varphi}$

$$\begin{cases} [\varphi(t,\vec{x}_1), \varphi(t,\vec{x}_2)] = 0 \\ [\dot{\varphi}(t,\vec{x}_1), \dot{\varphi}(t,\vec{x}_2)] = 0 \\ [\varphi(t,\vec{x}_1), \dot{\varphi}(t,\vec{x}_2)] = i\delta^{(3)}(\vec{x}_1 - \vec{x}_2) \end{cases} \longleftarrow$$

postulate based on causality

normal mode expansion

$$\varphi(\vec{x},t) = \underbrace{\int \frac{d^3\vec{k}}{(2\pi)^3}}_{\sum_{\vec{k}}} \left(a(\vec{k}) e^{-ikx} + \overset{a^+(\vec{k})}{b^+(\vec{k})} e^{ikx} \right) \underbrace{N_{\vec{k}}}_{\text{normal.}}$$

φ = real ; $b = a$

$K = (E_{\vec{k}}, \vec{k})$

$K^2 = m^2 \Rightarrow E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$

$\Big\downarrow$ QFT

$\hat{\varphi}$ | $\hat{a}(\vec{k})$ annihilate particle with mom $\vec{k}$, $E_{\vec{k}}$
| $\hat{a}^+(\vec{k})$ create particle with mom $\vec{k}$, $E_{\vec{k}}$

a, a^+ satisfy also com. relations

$$k \cdot x = k^0 x^0 - \vec{k} \cdot \vec{x} = E_{\vec{k}} t - \vec{k}\vec{x}$$

$$i\dot{\hat{\varphi}}(\vec{x},t) = \int \frac{d^3\vec{k}}{(2\pi)^3} N_{\vec{k}} \cdot E_{\vec{k}} \left(\hat{a}(\vec{k}) e^{-ikx} - \hat{a}^+(\vec{k}) e^{ikx} \right)$$

Let's consider

$$\bullet \underbrace{\int d^3\vec{x}\, e^{-i\vec{k}'\vec{x}} \hat{\varphi}(x)}_{} = N_{\vec{k}} \int d^3\vec{x} \int \frac{d^3\vec{k}}{(2\pi)^3} \Big(\underbrace{e^{i(\vec{k}-\vec{k}')\vec{x} - iE_{\vec{k}}t}}_{(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')} \cdot \hat{a}(\vec{k}) \cdot + \underbrace{e^{-i(\vec{k}+\vec{k}')x + iE_{\vec{k}}t}}_{(2\pi)^3 \delta^{(3)}(\vec{k}+\vec{k}')} a^+(\vec{k}) \Big)$$

$$= e^{-iE_{\vec{k}'}t} N_{\vec{k}'} \underline{a(\vec{k}')} + e^{iE_{-\vec{k}'}t} N_{-\vec{k}'} \underline{a^+(-\vec{k}')}$$

$$\bullet\ \int d^3\vec{x}\, e^{-i\vec{k}'\vec{x}}\, i\dot{\varphi}(x) = e^{-iE_{\vec{k}'}t} N_{\vec{k}'} E_{\vec{k}'}\, a(\vec{k}') - e^{iE_{-\vec{k}'}t} \cdot N_{-\vec{k}'} E_{-\vec{k}'}\, a^{\dagger}(-\vec{k}')$$

$\varphi = \varphi^{\dagger}$ real K.G. field

Finally we obtain:

$$a(\vec{k}) = \frac{1}{2E_{\vec{k}} N_{\vec{k}}} \int d^3\vec{x}\, e^{ikx} \left(E_{\vec{k}}\varphi + i\dot{\varphi}\right)$$

$$a^{\dagger}(\vec{k}) = \frac{1}{2E_{\vec{k}} N_{\vec{k}}} \int d^3\vec{x}\, e^{-ikx} \left(E_{\vec{k}}\varphi - i\dot{\varphi}\right)$$

$$[a(\vec{k}), a^{\dagger}(\vec{k}')] = \frac{1}{(2E_{\vec{k}} N_{\vec{k}})(2E_{\vec{k}'} N_{\vec{k}'})} \int d^3\vec{x} \int d^3\vec{x}'\, e^{ikx - ik'x'} \cdot$$

$x^0 = (x')^0$

$$\cdot\, \underbrace{\left[E_{\vec{k}}\varphi(x) + i\dot{\varphi}(x),\ E_{\vec{k}'}\varphi(x') - i\dot{\varphi}(x')\right]}_{-iE_{\vec{k}}\, i\delta^{(3)}(\vec{x}-\vec{x}') - iE_{\vec{k}'}\, i\delta^{(3)}(\vec{x}-\vec{x}')}$$

$$= \frac{1}{(2E_{\vec{k}} N_{\vec{k}})(\cancel{2E_{\vec{k}'}} N_{\vec{k}'})} \underbrace{(E_{\vec{k}} + E_{\vec{k}'})}_{\cancel{2E_{\vec{k}}}} \cdot \underbrace{\int d^3\vec{x}\, e^{i(E_{\vec{k}} - E_{\vec{k}'})t}\, e^{-i(\vec{k}-\vec{k}')\vec{x}}}_{(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$= \frac{1}{2E_{\vec{k}} N_{\vec{k}}^2} (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$N_{\vec{k}} = \frac{1}{\sqrt{2E_{\vec{k}}}} \quad \Rightarrow \quad \begin{cases} [a(\vec{k}), a^{\dagger}(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}') \\ [a(\vec{k}), a(\vec{k}')] = 0 \\ [a^{\dagger}(\vec{k}), a^{\dagger}(\vec{k}')] = 0 \end{cases}$$

covariant normal. condition

$$H = \int d^3\vec{x}\, \mathcal{H} = \int d^3\vec{x} \left(\frac{1}{2}(\dot{\varphi})^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} E_{\vec{k}} \left(a^{\dagger}(\vec{k})\, a(\vec{k}) + \frac{1}{2} \right)$$

$$\vec{P} = \int d^3\vec{x}\, \dot{\varphi}(-\vec{\nabla}\varphi) = \int \frac{d^3\vec{k}}{(2\pi)^3}\, \vec{k} \left(a^{\dagger}(\vec{k})\, a(\vec{k}) \right)$$

Single particle states

Define vacuum state

$|0\rangle$ state with no particles

$$\langle 0|0\rangle = 1$$

$$(2E_{\vec k})^{1/2}\langle 0|\,a(\vec k) = \langle \vec k|$$

$$a(\vec k)\,|0\rangle = 0$$

$$(2E_{\vec k})^{1/2}\,a^{+}(\vec k)\,|0\rangle = |\vec k\rangle$$

$$[a(\vec k), a^{+}(\vec k')] = (2\pi)^3\,\delta^{(3)}(\vec k-\vec k')$$

$$\langle \vec k|\vec k'\rangle = (2E_{\vec k})^{1/2}(2E_{\vec k'})^{1/2}\,\langle 0|\,a(\vec k)\,a^{+}(\vec k')\,|0\rangle$$

$$+\cancel{a^{+}(\vec k')\,a(\vec k)} - a^{+}(\vec k')\,a(\vec k)$$

$$= (2E_{\vec k})^{1/2}(2E_{\vec k'})^{1/2}\,\langle 0|\,[a(\vec k), a^{+}(\vec k')]\,|0\rangle$$

$$= (2E_{\vec k})(2\pi)^3\,\delta^{(3)}(\vec k-\vec k')$$

Lorentz invar.

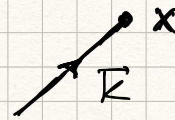
$$\varphi(x)\,|0\rangle = \int \frac{d^3\vec k'}{(2\pi)^3}\frac{1}{(2E_{\vec k'})^{1/2}}\left(e^{-ik'x}\,\overbrace{\cancel{a(\vec k')|0\rangle}}^{0} + e^{ik'x}\,\underbrace{a^{+}(\vec k')|0\rangle}_{\frac{1}{(2E_{\vec k'})^{1/2}}|\vec k'\rangle}\right)$$

$$= \int \frac{d^3\vec k'}{(2\pi)^3}\,\frac{e^{ik'x}}{2E_{\vec k'}}\,|\vec k'\rangle$$

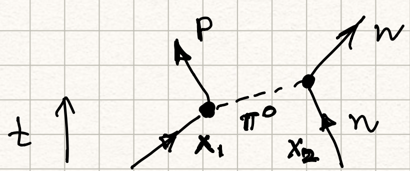
- $$\langle \vec k|\,\varphi(x)\,|0\rangle = \int \frac{d^3\vec k'}{\cancel{(2\pi)^3}}\,\frac{1}{\cancel{2E_{\vec k'}}}\,e^{ik'x}\,\underbrace{\langle \vec k|\vec k'\rangle}_{\cancel{(2\pi)^3}\,\cancel{2E_{\vec k}}\,\delta^{(3)}(\vec k-\vec k')}$$

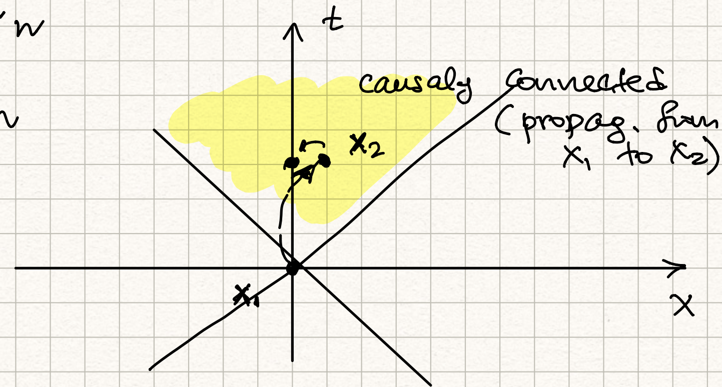
$$= e^{ikx}$$

- $\langle 0 | \varphi(x) | \vec{k} \rangle = e^{-ikx}$



nucleus: p, n interaction with exchange of π^0 (spin-0)





4) Feynman propagator for Klein Gordon field

NOT equal time

$$\langle 0 | \underline{\varphi(x_2)\varphi(x_1)} | 0 \rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 (2E_{\vec{k}})^{1/2}} \int \frac{d^3\vec{k}'}{(2\pi)^3 (2E_{\vec{k}'})^{1/2}} \cdot$$

$$\cdot \langle 0 | \left(e^{-ikx_2} a(\vec{k}) + \underbrace{\cancel{e^{ikx_2} a^{*}(\vec{k})}}_{0} \right) \left(\underbrace{\cancel{e^{-ik'x_1} a(\vec{k}')}}_{0} + e^{ik'x_1} a^{+}(k') \right) | 0 \rangle =$$

$$(a(\vec{k}) | 0 \rangle = 0)^{+}$$

$$\langle 0 | a^{+}(\vec{k}) = 0$$

$$-a^{+}(\vec{k}) a(\vec{k}) | 0 \rangle = 0$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 (2E_{\vec{k}})^{1/2}} \int \frac{d^3\vec{k}'}{(2\pi)^3 (2E_{\vec{k}'})^{1/2}} e^{-ikx_2} e^{ik'x_1} \langle 0 | \underbrace{a(\vec{k})\, a^{+}(\vec{k}')}_{[\, a(\vec{k}),\, a^{+}(\vec{k}') \,]} | 0 \rangle$$

$$\underbrace{[\, a(\vec{k}),\, a^{+}(\vec{k}') \,]}_{(2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$= \underline{\int \frac{d^3\vec{k}}{(2\pi)^3 2E_{\vec{k}}}} e^{-ik(x_2-x_1)} \equiv i\Delta(x_2 - x_1)$$