Lecture 2 (22 April 2020)

I Klein Gordon theory

2) Classical field theory

real fields (for simplicity) $\xrightarrow{\text{extend}}$ complex fields

$$\varphi_r(x) \equiv \varphi_r(t, \vec{x}) \qquad r = 1 \quad \text{scalar field}$$

$$A^\mu(t, \vec{x}) \qquad r = 1, 2, 3, 4 \quad \text{vector field}$$

Lagrangian density function

$$\mathcal{L}(\varphi_r(x), \partial^\mu \varphi_r(x)) \qquad \text{(covar.)}$$

$$L(t) = \int d^3\vec{x}\; \mathcal{L}(\varphi_r(x), \partial^\mu \varphi_r(x)) \qquad \text{(not covar.)}$$

Variational principle

$$S \equiv \int_\Omega d^4x\; \mathcal{L}(\varphi_r(x), \partial^\mu \varphi_r(x))$$

$$\varphi_r \to \varphi_r + \underbrace{\delta\varphi_r}_{\text{variation}} \; : \quad \delta S = 0$$

Require: on the boundary $\delta\varphi_r \big|_{\Gamma(\Omega)} = 0$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi_r}\, \delta\varphi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)}\, \delta(\partial_\mu \varphi_r) \right)$$

$$+ \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \right) \delta\varphi_r - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \right) \delta\varphi_r$$

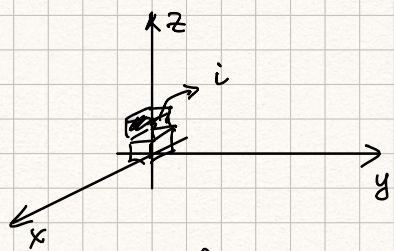
$$= \int d^4x \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi_r}\, \delta\varphi_r - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \right) \delta\varphi_r \right) + \overset{0}{\cancel{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)}\, \delta\varphi_r \right)}} \right\}$$

for arbitrary $\delta\varphi_r$: $\delta S = 0$

$$\frac{\partial \mathcal{L}}{\partial \varphi_r} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} = 0 \qquad \text{Euler Lagrange eq.}$$

Hamiltonian field theory

discretize the space



$$\varphi_\eta(x) \equiv \varphi_\eta(t,\vec{x}) \xrightarrow{\text{discr.}} \varphi_\eta(t,i)$$

Lagrangian function $L(t)$

$$L(t) = \int d^3x \; \underbrace{\mathcal{L}(\varphi_\eta, \partial_\mu \varphi_\eta)} \xrightarrow{\text{discr}} \sum_i \overbrace{(\delta \vec{x}_i)}^{\text{volume element}} \mathcal{L}(\varphi_\eta(t,i), \dot{\varphi}_\eta(t,i))$$

$\delta \vec{x}_i \to 0$ continuous limit

$$\dot{\varphi}_\eta \equiv \frac{\partial \varphi_\eta}{\partial t}$$

$\varphi_\eta(t,i) \equiv q_{\eta,i}(t)$ generalized coordinates

Conjugate (general) momenta

$$p_{\eta,i} = \frac{\partial L}{\partial \dot{q}_{\eta,i}} \equiv \frac{\partial L}{\partial \dot{\varphi}_\eta(t,i)} = \underbrace{\delta \vec{x}_i} \; \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\eta(t,i)}}$$

$$\Pi_\eta(t,i) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\eta(t,i)} \xrightarrow{\text{continuum}} \Pi_\eta(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_\eta(x)}$$

$$H = p\dot{q} - L$$

$$H = \sum_i p_{\eta,i} \, \dot{q}_{\eta,i} - L$$

$$= \sum_i (\delta \vec{x}_i) \left(\Pi_\eta(t,i) \dot{\varphi}_\eta(t,i) - \mathcal{L} \right)$$

$$\xrightarrow{\text{continuum}} \int d^3\vec{x} \left(\underbrace{\Pi_\eta(x) \dot{\varphi}_\eta(x) - \mathcal{L}(x)}_{\mathcal{H}(x)} \right)$$

$\mathcal{H}(x)$ Hamiltonian density

Example: Hamiltonian for K.G. field

$$(\partial_\mu \partial^\mu + m^2)\,\varphi(x) = 0 \qquad \mathcal{L} = \,?$$

E.L. $$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2$$

Proof

$$\frac{\partial \mathcal{L}}{\partial \varphi} = - m^2 \varphi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \partial^\mu \varphi$$

$$\partial_\mu \partial^\mu \varphi + m^2 \varphi = 0$$

$$\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) = \frac{1}{2}\left(\left(\frac{\partial \varphi}{\partial t}\right)^2 - (\vec{\nabla}\varphi)^2\right)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} \; ; \; \frac{\partial \mathcal{L}}{\partial(\vec{\nabla}\varphi)} = -\vec{\nabla}\varphi$$

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$$

$$\mathcal{H}_{KG} = \Pi \cdot \dot{\varphi} - \mathcal{L} = (\dot{\varphi})^2 - \mathcal{L} = \frac{1}{2}\left((\dot{\varphi})^2 + (\vec{\nabla}\varphi)^2 + m^2 \varphi^2\right)$$

Noether's theorem

SYMMETRIES $\leftrightarrow$ CONSERVATION LAW

$\mathcal{L}$ encoded

- continuous transformation of the fields

$$\varphi_r(x) \rightarrow \varphi'_r(x) = \varphi_r(x) + \delta\varphi_r(x)$$

$$\delta \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \varphi_r}}_{\text{E.L. eq}} \delta\varphi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta(\partial_\mu \varphi_r) =$$

$$= \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)}\right) \delta\varphi_r + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \partial_\mu(\delta\varphi_r)$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r \right)$$

above transformation is a symmetry transformation if it leaves the action invariant

$$S = \int d^4x\, \mathcal{L} = \text{invar} \qquad (\delta S = 0)$$

- $\mathcal{L}$ is invariant $(\delta\mathcal{L} = 0)$ $\quad \underline{\mathcal{L}' = \mathcal{L}}$
- $\mathcal{L}$ changes by a full <u>divergence</u>

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \underbrace{\partial_\mu J^\mu}_{\delta\mathcal{L}} \qquad \int d^4x\, \partial_\mu J^\mu \to J^\mu\big|_{\Gamma} = 0$$

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r \right) \Rightarrow \partial_\mu \Big(\overbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi - J^\mu}^{j^\mu} \Big) = 0$$

$$\boxed{j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_r)} \delta\varphi_r - J^\mu} \Rightarrow j^0 = \Pi_r \delta\varphi_r - J^0$$

$$\boxed{\underbrace{\partial_\mu j^\mu = 0}} \leftarrow \quad j^\mu = \text{conserved current}$$

$$\frac{d}{dt} j^0 + \vec{\nabla} \vec{j} = 0 \qquad \int d^3\vec{x}\, (\ldots)$$

$$\frac{d}{dt} \underbrace{\int d^3\vec{x}\, j^0(t,\vec{x})}_{Q} = - \underbrace{\int d^3\vec{x}\, \cancel{\vec{\nabla} \vec{j}(t,\vec{x})}}_{0}$$

Conserved charge

$$\boxed{Q = \int d^3\vec{x}\, j^0(t,\vec{x}) \,; \quad \frac{dQ}{dt} = 0}$$

EXAMPLE: symmetry under space-time translation

$$x^\mu \to x'^\mu = x^\mu + \underbrace{a^\mu}$$

constant transl
(small, continuous)
↓
extend to any

$$\varphi(x) \to \underline{\underline{\varphi'(x)}} = \varphi(x) + \delta\varphi(x)$$

$\varphi'(x') = \varphi(x)$ transformed field at shifted position = original field at original position

(symmetry)

$J^\mu = ? \leftarrow \delta\mathcal{L}$

variation of the field

$$\underline{\underline{\delta\varphi(x)}} = \varphi'(x) - \overbrace{\varphi(x)}^{\varphi'(x')^{\,x+a}} = \varphi'(x) - \underline{\varphi'(x+a)}$$

$$= -\left(a^\mu \partial_\mu \varphi\right) + \underline{\underline{O(a^2)}}$$

a is small
Taylor

$f(x+a) - f(x) = a \cdot f'(x)$

$$\underline{\underline{\delta\mathcal{L}}} = -a^\mu \partial_\mu \mathcal{L} = \partial_\mu J^\mu$$

Proof:

$$\mathcal{L} = \underline{\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi)} - \underline{\frac{1}{2}m^2\varphi^2}$$

$\varphi \to \varphi' = \varphi - a^\nu(\partial_\nu\varphi) + O(a^2)$ $\quad a = \text{small}$

$$\partial_\mu\varphi \to \partial_\mu\varphi' = \partial_\mu\varphi - a^\nu(\partial_\mu\partial_\nu\varphi) + O(a^2)$$

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} - a^\nu(\partial_\mu\partial_\nu\varphi)\partial^\mu\varphi + \cancel{O(a^2)}$$
$$+ a^\nu m^2(\partial_\nu\varphi)\varphi + O(a^2)$$

$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$

$$\delta\mathcal{L} = -a^\nu(\partial_\nu\mathcal{L})$$

$$\boxed{J^\mu = -a^\mu\mathcal{L}}$$

Conserved current $\quad \partial_\mu j^\mu = 0$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi - J^\mu$$

$$= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} a_\nu \partial^\nu \varphi + \underbrace{a^\mu \mathcal{L}}_{a_\nu g^{\mu\nu}} = a_\nu \left(-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi + \mathcal{L} g^{\mu\nu} \right)$$

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L} \right)}_{T^{\mu\nu}} \cancel{a_\nu} = 0 \qquad \forall a_\nu$$

consequence of space/time transl.

$\downarrow$

conserved energy/mom tensor

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}$$

$$\partial_\mu j^\mu = 0 \rightarrow \frac{dj^0}{dt} - \cancel{\vec{\nabla} \vec{j}} = 0 \qquad \int d^3x$$

$$\frac{d}{dt} Q = 0$$

$$Q = \int d^3\vec{x}\, j^0$$

"0" component

$$p^\nu = \int d^3\vec{x}\, T^{0\nu}$$

$$= \int d^3\vec{x} \left(\Pi \partial^\nu \varphi - g^{0\nu} \mathcal{L} \right) \qquad \text{4-mom.}$$

if we take $\nu = 0$

$$p^0 = \int d^3\vec{x} \, (\underbrace{\Pi \dot{\varphi} - \mathcal{L}}_{\mathcal{H}}) = \underline{\underline{H}}$$

$$\vec{p} = \int d^3\vec{x} \left(\Pi \cdot (-\vec{\nabla}\varphi) \right) = -\int d^3\vec{x}\, \Pi \vec{\nabla} \varphi$$