2 problems. Total number of points: 26.
Grading: 20+ excellent, 15+ good, below 10 is incomplete.
Deadline: Thursday, January 30.

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1. **Tree-level unitarity in the Georgi-Glashow SU(2) model of electroweak interactions (1972)**

   (a) {2pts} Consider the SU(2) Yang-Mills theory with a real scalar field ϕ_i, $i = 1..3$, in adjoint representation of SU(2)

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{a\,\mu\nu} + \frac{1}{2}\mathcal{D}^{ac}_\mu\phi^c \cdot \mathcal{D}^{ad\,\mu}\phi^d - V(\phi), \tag{1}$$

   where Higgs potential $V(\phi) = -\mu^2\phi\cdot\phi + \lambda(\phi\cdot\phi)^2$, the covariant derivative $\mathcal{D}^{ac}_\mu = \delta^{ac}\partial_\mu + g\epsilon^{abc}\mathcal{A}^b_\mu$ (ϵ^{abc} is the Levi-Civita symbol), and $a/b/c = 1,2,3$. Choose the VEV of ϕ to reproduce the mass term in Eq.(4) from the problem sheet 4

$$\mathcal{L}_M = \frac{M_W^2}{2}[\,(\mathcal{A}^1_\mu)^2 + (\mathcal{A}^2_\mu)^2\,]. \tag{2}$$

   Find the mass M_W in terms of parameters of the Higgs potential.

   Following H. Georgi and S. L. Glashow, let us identifyA_3^μ as the photon field, A_1^μ with A_2^μ are gathered into complex vector fields $W^\pm$ (as in 1.(c) of the problem sheet 4) and describe weak interactions.

   In order to check the tree-level unitarity of the theory, one should define the behavior of the tree-level amplitudes at high energies. Consider the elastic scattering of W^+ on W^- at the tree level in the center-of-mass frame. It can be described by 5 Feynman diagrams: contact term, s- and t-channel photon exchange and s- and t-channel Higgs scalar exchange.

   (b) {2pts} Using Feynman rules for Yang-Mills theories write the amplitudes for the contact term $\mathcal{M}_c$, for the s-channel photon exchange $\mathcal{M}_s$ and for the t-channel photon exchange $\mathcal{M}_t$ in terms of momenta $p_1^\mu..p_4^\mu$ and corresponding polarization vectors $\varepsilon_1^\mu..\varepsilon_4^\mu$ of W's. Check yourself by the following expressions:

$$\mathcal{M}_c = g^2\Big[2\varepsilon_1\cdot\varepsilon_4^*\,\varepsilon_2\cdot\varepsilon_3^* - \varepsilon_1\cdot\varepsilon_3^*\,\varepsilon_2\cdot\varepsilon_4^* - \varepsilon_1\cdot\varepsilon_2\,\varepsilon_3^*\cdot\varepsilon_4^*\Big], \tag{3}$$

$$\mathcal{M}_s = \frac{g^2}{s}\Big[(p_1-p_2)^\mu\,\varepsilon_1\cdot\varepsilon_2 + 2\varepsilon_2^\mu\,p_2\cdot\varepsilon_1 - 2\varepsilon_1^\mu\,p_1\cdot\varepsilon_2\Big]$$
$$\times\Big[(p_3-p_4)_\mu\,\varepsilon_3^*\cdot\varepsilon_4^* + 2\varepsilon^*_{4\,\mu}\,p_4\cdot\varepsilon_3^* - 2\varepsilon^*_{3\,\mu}\,p_3\cdot\varepsilon_4^*\Big], \qquad (4)$$
$$\mathcal{M}_t = -\frac{g^2}{t}\Big[(p_1+p_3)^\mu\,\varepsilon_1\cdot\varepsilon_3^* - 2\varepsilon_1^\mu\,p_1\cdot\varepsilon_3^* - 2\varepsilon_3^{*\mu}\,p_3\cdot\varepsilon_1\Big]$$
$$\times\Big[(p_2+p_4)_\mu\,\varepsilon_2\cdot\varepsilon_4^* - 2\varepsilon^*_{4\,\mu}\,p_4\cdot\varepsilon_2 - 2\varepsilon_{2\,\mu}\,p_2\cdot\varepsilon_4^*\Big], \qquad (5)$$

Considered amplitudes will have the worst behavior when one chooses the all polarization vectors of $W^\pm$-bosons to be longitudinal due to the scalar products of only such vectors can be proportional to s when $s\to\infty$.

(c) {5pts} Show that when the all polarization vectors $\varepsilon_1^\mu..\varepsilon_4^\mu$ are longitudinal and $s\to\infty$, the sum of $\mathcal{M}_c$, $\mathcal{M}_s$ and $\mathcal{M}_t$ has the following leading-order divergent term:

$$\mathcal{M}_c+\mathcal{M}_s+\mathcal{M}_t|_{s\to\infty} = \frac{g^2 s}{2M_W^2}(1+\cos\theta)+O(s^0), \qquad (6)$$

where θ is the scattering angle.

Hint: use $t=-\frac{s}{2}\left[1-\frac{4M_W^2}{s}\right](1-\cos\theta)$ and Mathematica if necessary.

(d) {2pts} Define the coupling of Higgs scalar to the $W^\pm$ in terms of coupling g and mass M_W. Determine the corresponding Feynman rule.

(e) {3pts} Calculate s- and u-channel amplitudes with Higgs scalar exhange. Show that when the all polarization vectors $\varepsilon_1^\mu..\varepsilon_4^\mu$ are longitudinal and $s\to\infty$, the leading-order term of these amplitudes cancels (6) explicitly, so the total amplitude does not contain any divergent term at high energies. Hence, the tree-level unitarity is fulfilled.

2. **Muon lifetime in the Fermi theory**

Decay of an ensemble of unstable particles obeys an exponential law

$$N(t) = N(0)\, e^{-t/\tau}\,, \tag{7}$$

with t being the time and τ being the particle lifetime. Equivalently, one can use the decay width Γ related to the lifetime as:

$$\Gamma = \frac{\hbar}{\tau} \tag{8}$$

(a) {9 pts} Show that the width of the muon decay in Fermi theory, to leading order in G_F and neglecting the electron and neutrino masses, is given by:

$$\Gamma(\mu \to e + \nu_e + \nu_\mu) = \frac{G_F^2 m_\mu^5}{192\pi^3} \tag{9}$$

(b) {1 pts} Calculate the corresponding lifetime of the muon and compare it with the experimental number:

$$\tau_\mu = (2.19703 \pm 0.00004)\, 10^{-6}\, s. \tag{10}$$

(c) {2 pts} Discuss the possible corrections which should improve the agreement of your calculation with experiment.