# PATH INTEGRALS IN QUANTUM MECHANICS

1) DERIVATION OF PATH INTEGRAL IN 1D
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3) EXAMPLE IN 3D : AHARONOV - BOHM EFFECT
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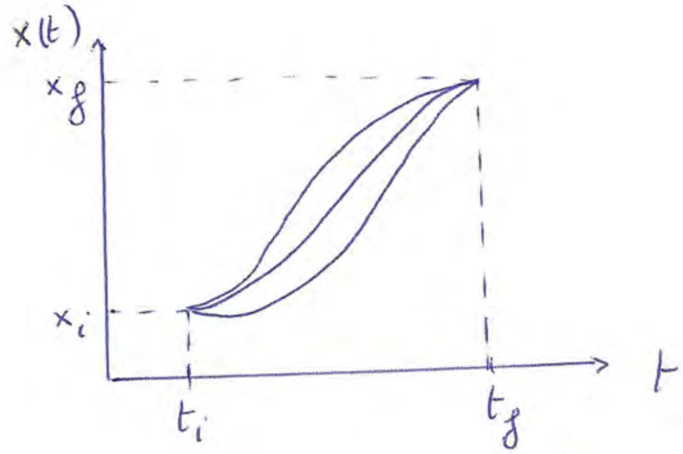
# 1) DERIVATION OF PATH INTEGRAL IN 1D

## • CLASSICAL ACTION




ACTION $\quad S[x(t)] = \int_{t_i}^{t_f} dt \; L(x(t), \dot{x}(t))$

CONSIDER VARIATIONS OF $S$ BY $\delta x(t)$

SUCH THAT $\quad \delta x(t_i) = 0 \;, \quad \delta x(t_f) = 0$

$\qquad\qquad x(t_i) = x_i \;, \quad x(t_f) = x_f$

$\Longrightarrow$ VARIATIONAL PRINCIPLE (PRINCIPLE OF LEAST ACTION)

CLASSICAL PATH MINIMIZES $S$

i.e. $\quad \underline{\underline{\delta S = 0}}$

$\Updownarrow$

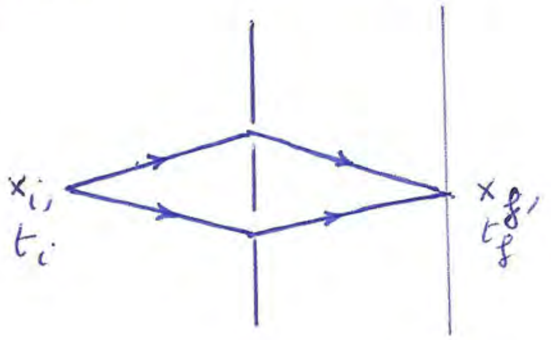
EULER-LAGRANGE EQ.

$$\underline{\underline{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0.}}$$

$\hookrightarrow$ SOLUTION GIVES CLASSICAL PATH

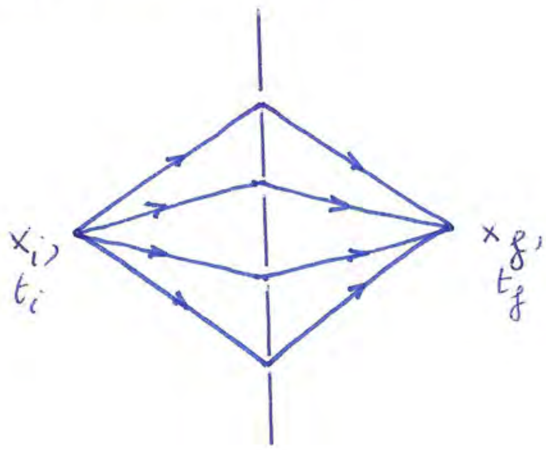
## • TRANSITION AMPLITUDE IN QUANTUM MECHANICS

↳ CONSIDER 2-SLIT EXP.



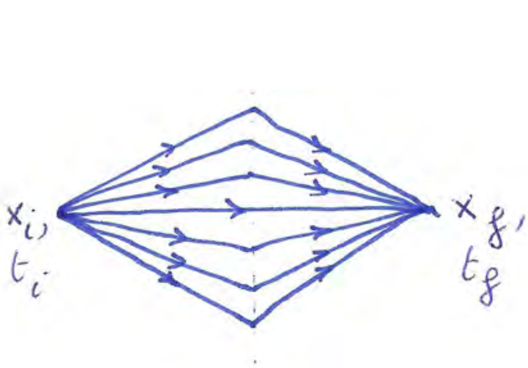

AT POSITION $x_f$ & TIME $t_f$
QM AMPLITUDE IS OBTAINED
AS SUM OF AMPLITUDE
FOR 2 PATHS
(INTERFERE)

↳ MAKE MORE SLITS




4 SLITS
QM AMPLITUDE AT $x_f$, $t_f$
OBTAINED AS SUM
OVER 4 PATHS

↳ MAKE INFINITE # SLITS ↷ INTERMEDIATE SCREEN DISAPPEARS




QM AMPLITUDE AT $x_f$, $t_f$
OBTAINED AS SUM
OVER ALL POSSIBLE PATHS

↳ CONSIDER POSITION OPERATOR IN SCHRÖDINGER PICTURE OF QM

$$\hat{X}_S \; |x\rangle_S = x \; |x\rangle_S$$

TIME-INDEP. OPERATOR IN SCHRÖDINGER PICTURE (S)

EIGENVALUE

TIME-INDEP. EIGENSTATES

↳ IN HEISENBERG PICTURE (H): OPERATOR & EIGENSTATES BECOME TIME DEPENDENT

$$\hat{X}_H(t) \equiv e^{\frac{i}{\hbar}\hat{H}t} \, \hat{X}_S \, e^{-\frac{i}{\hbar}\hat{H}t}$$

$\hat{H}$ HAMILTONIAN

$$|x,t\rangle \equiv e^{\frac{i}{\hbar}\hat{H}t} \; |x\rangle_S$$

NOTE $|x, t=0\rangle = |x\rangle_S = |x\rangle$ IN FOLLOWING

$$\Downarrow$$

$$\hat{X}_H(t) \; |x,t\rangle = x \; |x,t\rangle$$

↳ TRANSITION AMPLITUDE (PROPAGATOR)

$$\boxed{\langle x_f, t_f \,|\, x_i, t_i \rangle \equiv K(x_f, t_f; x_i, t_i)}$$

PROBABILITY AMPLITUDE THAT SYSTEM WHICH IS IN EIGENSTATE $|x_i, t_i\rangle$ AT TIME $t_i$ WILL BE IN EIGENSTATE $|x_f, t_f\rangle$ AT TIME $t_f$

↳ INSERT COMPLETE SET OF STATES OF H

$$K(x_f, t_f; x_i, t_i)$$

$$= \langle x_f, t_f | x_i, t_i \rangle$$

$$= \langle x_f | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | x_i \rangle$$

$$= \sum_n \sum_{n'} \langle x_f | \psi_n \rangle \langle \psi_n | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | \psi_{n'} \rangle \langle \psi_{n'} | x_i \rangle$$

$|\psi_n\rangle$ : EIGENSTATES OF $\hat{H}$

$$\hat{H} | \psi_n \rangle = E_n | \psi_n \rangle$$

$$\langle \psi_n | \psi_{n'} \rangle = \delta_{nn'}$$

$$= \sum_n \underbrace{\langle x_f | \psi_n \rangle}_{\psi_n(x_f)} e^{-\frac{i}{\hbar} E_n (t_f - t_i)} \underbrace{\langle \psi_n | x_i \rangle}_{\psi_n^*(x_i)}$$

$$= \sum_n e^{-\frac{i}{\hbar} E_n (t_f - t_i)} \psi_n(x_f) \psi_n^*(x_i)$$

FOURIER ANALYZING GIVES EIGENVALUES $E_n$ OF $\hat{H}$

& FOURIER COEFFICIENTS EIGENSTATES $|\psi_n\rangle$

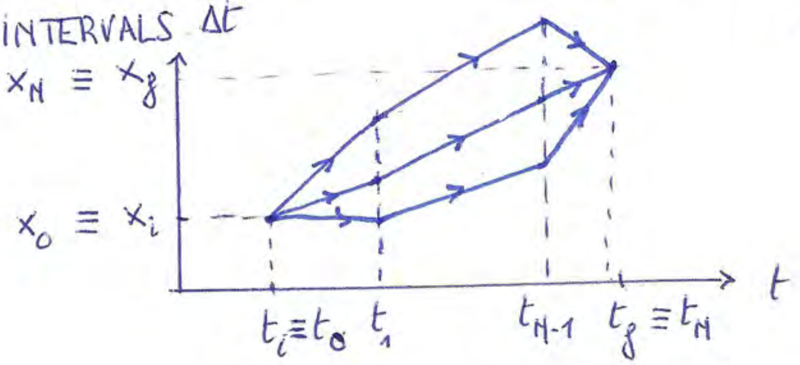
∴ PROPAGATOR CONTAINS ALL DYNAMICAL INFORMATION ON QUANTUM SYSTEM

## • TRANSITION AMPLITUDE AS PATH INTEGRAL

R.P. FEYNMAN (1948)

↳ BREAK UP TIME INTERVAL $t_f - t_i$ INTO N INFINITESIMAL INTERVALS $\Delta t$

$$t_k = t_i + k\,\Delta t$$




+ USE COMPLETENESS OF EIGENSTATES AT INTERMEDIATE $t_k$

$$\int_{-\infty}^{+\infty} dx_k \, |x_k, t_k\rangle\langle x_k, t_k| = \mathbb{1}$$

↳ $K(x_f, t_f\,;\, x_i, t_i)$

$$= \prod_{k=1}^{N-1} \int dx_k \, \langle x_N, t_N | x_{N-1}, t_{N-1}\rangle \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2}\rangle \cdots$$

$$\cdots \langle x_2, t_2 | x_1, t_1\rangle \langle x_1, t_1 | x_0, t_0\rangle$$

↳ FOR INFINITESIMAL INTERVAL

$$\langle x_{k+1}, t_{k+1} | x_k, t_k\rangle$$

$$= \langle x_{k+1} | e^{-\frac{i}{\hbar}\hat{H}(t_{k+1}-t_k)} | x_k\rangle$$

$$= \langle x_{k+1} | e^{-\frac{i}{\hbar}\hat{H}(\Delta t)} | x_k\rangle$$

INSERT COMPLETE SET OF MOMENTUM EIGENSTATES

$$\hat{P} \, | p_k \rangle = p_k \, | p_k \rangle$$

$$\langle p_k | p_{k'} \rangle = \delta ( p_{k'} - p_k )$$

$$\int_{-\infty}^{+\infty} dp_k \, | p_k \rangle \langle p_k | = \mathbb{1}$$

$$\therefore \quad \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$$

$$= \int dp_k \, \langle x_{k+1} | p_k \rangle \langle p_k | \, e^{-\frac{i}{\hbar} \hat{H} \Delta t} \, | x_k \rangle$$

$\downarrow$ $\qquad \hat{H}$ DEPENDS ON $\hat{x}$ & $\hat{P}$

$$= \int dp_k \, \langle x_{k+1} | p_k \rangle \langle p_k | x_k \rangle \, e^{-\frac{i}{\hbar} H (x_k, p_k) \Delta t}$$

NOTE : THIS IS THE HAMILTONIAN FUNCTION NOT OPERATOR ANY MORE

e.g. $H(x, p) = \frac{p^2}{2m} + V(x)$

$\hookrightarrow$ USE

$$\int dp_k \, \langle x_{k+1} | p_k \rangle \langle p_k | x_k \rangle = \langle x_{k+1} | x_k \rangle$$

$$= \delta ( x_{k+1} - x_k )$$

$$= \frac{1}{2\pi\hbar} \int dp_k \, e^{+\frac{i}{\hbar} p_k ( x_{k+1} - x_k )}$$

$$\langle x_{k+1} | p_k \rangle = \frac{1}{\sqrt{2\pi\hbar}} \, e^{\frac{i}{\hbar} p_k x_{k+1}}$$

$$\langle p_k | x_k \rangle = \frac{1}{\sqrt{2\pi\hbar}} \, e^{-\frac{i}{\hbar} p_k x_k}$$

$$\hookrightarrow \langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$$

$$= \int \frac{dp_k}{2\pi\hbar} \, e^{\frac{i}{\hbar} \left[ p_k (x_{k+1} - x_k) - H(x_k, p_k) \cdot \Delta t \right]}$$

$\downarrow$ FOR $\Delta t$ INFINITESIMAL

$$x_{k+1} - x_k = \dot{x}_k (\Delta t)$$

$$\boxed{\langle x_{k+1}, t_{k+1} | x_k, t_k \rangle = \int \frac{dp_k}{2\pi\hbar} \, e^{\frac{i}{\hbar} \left[ p_k \dot{x}_k - H(x_k, p_k) \right] \Delta t}}$$

## ↳ FINITE TRANSITION AMPLITUDE

$$K(x_f, t_f ; x_i, t_i)$$

$$= \prod_{k=1}^{N-1} \int dq_k \prod_{k=0}^{N-1} \int \frac{dp_k}{2\pi\hbar}$$

$$\cdot \exp\left\{ \frac{i}{\hbar} \underbrace{\sum_{k=0}^{N-1} \left[ p_k \dot{x}_k - H(x_k, p_k) \right] \Delta t}_{\xrightarrow[\Delta t \to 0]{} \int_{t_i}^{t_f} dt \left[ p\dot{x} - H(x,p) \right]} \right\}$$

## ↳ PATH INTEGRAL

$$\boxed{K(x_f, t_f ; x_i, t_i) = \int \mathcal{D}x(t)\, \mathcal{D}p(t)\, \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \left[ p\dot{x} - H(x,p) \right] \right\}}$$

WITH PATH INTEGRAL 'MEASURES'

$$\mathcal{D}x(t) \equiv \lim_{N\to\infty} \prod_{k=1}^{N-1} dx(t_k)$$

$$\mathcal{D}p(t) \equiv \lim_{N\to\infty} \prod_{k=0}^{N-1} \frac{dp(t_k)}{2\pi\hbar}$$

WITH $x(t_i) = x_i$ AND $x(t_f) = x_f$

NOTE : IN EXPONENTIAL WE HAVE CLASSICAL ACTION FUNCTION OF SYSTEM IN TERMS OF HAMILTONIAN

$$\boxed{S = \int_{t_i}^{t_f} dt \left[ p \dot{x} - H(x, p) \right]}$$

$\therefore$ PATH INTEGRAL IS

→ FUNCTIONAL INTEGRAL OVER ALL POSSIBLE TRAJECTORIES IN PHASE SPACE OF SYSTEM

→ WEIGHTED BY $e^{\frac{i}{\hbar} S}$ WITH $S$ THE HAMILTONIAN ACTION

- **PATH INTEGRAL IN TERMS OF LAGRANGIAN ACTION**

FOR $H(x,p) = \frac{p^2}{2m} + V(x)$

WE CAN PERFORM THE $p_k$ INTEGRATIONS (FORMALLY)

↳ $\langle x_{k+1}, t_{k+1} | x_k, t_k \rangle$

$$= \int \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar}\left[p_k \dot{x}_k - H(x_k, p_k)\right]\Delta t}$$

$$= e^{-\frac{i}{\hbar} V(x_k)\Delta t} \int \frac{dp_k}{2\pi\hbar} e^{\frac{i}{\hbar}\left[-\frac{p_k^2}{2m} + p_k \dot{x}_k\right]\Delta t}$$

$$= e^{-\frac{i}{\hbar} V(x_k)\Delta t} e^{+\frac{i}{\hbar}\frac{1}{2} m \dot{x}_k^2} \int \frac{dp_k}{2\pi\hbar} e^{-\frac{i\Delta t}{\hbar 2m}(p_k - m\dot{x}_k)^2}$$

$\Big\downarrow$ ANALYTICAL CONTINUATION TO IMAG. TIME

$\Delta\tau = i\Delta t$ REAL

\+ GAUSSIAN INTEGRAL

$$\int_{-\infty}^{+\infty} dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$= e^{+\frac{i}{\hbar}\left[\frac{1}{2} m \dot{x}_k^2 - V(x_k)\right]\Delta t} \cdot \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{1/2}$$

$$= \left(\frac{m}{2\pi\hbar i\Delta t}\right)^{1/2} e^{\frac{i}{\hbar} L(x_k, \dot{x}_k)\Delta t}$$

$$\boxed{\begin{aligned}&\langle x_{k+1}, t_{k+1} \mid x_k, t_k \rangle \\ &= \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} e^{\frac{i}{\hbar} L(x_k, \dot{x}_k) \Delta t}\end{aligned}}$$

WITH LAGRANGIAN FUNCTION

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - V(x)$$

↳ FINITE TRANSITION AMPLITUDE

$$K(x_f, t_f ; x_i, t_i)$$

$$= \lim_{N \to \infty} \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{N/2} \int_{-\infty}^{+\infty} \prod_{k=1}^{N-1} dx_k \, e^{\frac{i}{\hbar} \sum_{k=0}^{N-1} L(x_k, \dot{x}_k) \Delta t}$$

$$\Downarrow$$

$$\boxed{\begin{aligned}&K(x_f, t_f ; x_i, t_i) \\ &= \int \widetilde{\mathcal{D}} x(t) \, \exp\left\{ \frac{i}{\hbar} \int_{t_i}^{t_f} dt \, L(x, \dot{x}) \right\} \\ &= \int \widetilde{\mathcal{D}} x(t) \, \exp\left\{ \frac{i}{\hbar} S \right\}\end{aligned}}$$

WITH

$$\widetilde{\mathcal{D}} x(t) \equiv \lim_{N \to \infty} \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{N/2} \prod_{k=1}^{N-1} dx(t_k)$$

PATH INTEGRAL

↳ SUM OVER ALL PATHS $x(t)$

↳ EACH PATH WEIGHTED WITH $e^{\frac{i}{\hbar}S}$

↳ CLASSICAL LIMIT $\hbar \to 0$

$e^{\frac{i}{\hbar}S}$ OSCILLATES RAPIDLY WHEN $\hbar \to 0$

ONLY PATH WHICH MAKES ACTION $S$ STATIONARY CONTRIBUTES TO PATH INTEGRAL

$$\delta S[x] = 0$$

$$\Downarrow$$

$$x(t) = x_{cl}(t)$$

CLASSICAL PATH

(SOLUTION OF EULER - LAGRANGE EQ.)

- EQUIVALENCE WITH SCHRÖDINGER EQUATION

↳ WAVEFUNCTION

$$\psi(x_f, t_f) = \langle x_f, t_f \,|\, \psi(t=0) \rangle$$

$$= \langle x_f \,|\, \underbrace{e^{-\frac{i}{\hbar}\hat{H} t_f} \,|\, \psi(t=0) \rangle}_{|\psi(t_f)\rangle}$$

$$= \langle x_f \,|\, \psi(t_f) \rangle$$

↳ CONNECTION WITH TRANSITION AMPLITUDE

$$\psi(x_f, t_f) = \int dx_i \, \langle x_f, t_f \,|\, x_i, t_i \rangle \langle x_i, t_i \,|\, \psi(t=0) \rangle$$

$$= \int dx_i \, K(x_f, t_f ; x_i, t_i) \, \psi(x_i, t_i)$$

↳ CONSIDER INFINITESIMAL TIMESTEP

$$t_i = t$$
$$t_f = t + \Delta t \qquad (\Delta t \to 0)$$

$$K(x_f, t+\Delta t ; x_i, t) = \left( \frac{m}{2\pi\hbar i \Delta t} \right)^{1/2} e^{\frac{i}{\hbar} L\left(\frac{x_i + x_f}{2}, \frac{x_f - x_i}{\Delta t}\right)\Delta t}$$

$$\psi(x_f, t+\Delta t) = \int dx_i \, K(x_f, t+\Delta t ; x_i, t) \, \psi(x_i, t)$$

↳ CONSIDER PARTICLE MOVING IN 1D

$$\underline{\underline{L = \frac{1}{2} m \dot{x}^2 - V(x,t)}}$$

⇝ $K(x_f, t+\Delta t\,;\, x_i, t)$

$$= \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2\Delta t}(x_f - x_i)^2}\, e^{-\frac{i}{\hbar}\Delta t\, V\left(\frac{x_i + x_f}{2}, t\right)}$$

$x_f \equiv x$

$x_i \equiv y$

↓

$$\rightsquigarrow \psi(x, t+\Delta t) = \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{1/2} \int_{-\infty}^{+\infty} dy\, e^{-\frac{m}{2\hbar i \Delta t}(x-y)^2}\, e^{-\frac{i}{\hbar}\Delta t\, V\left(\frac{x+y}{2}, t\right)} \cdot \psi(y,t)$$

ONLY REGION $y \approx x$ CONTRIBUTES MAINLY TO INTEGRAL

$$y = x + \eta \qquad (\text{ONLY REGION AROUND } \eta \approx 0)$$

CHANGE INTEGRATION VARIABLE

$$\psi(x, t+\Delta t) = \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta\, e^{-\frac{m\eta^2}{2\hbar i \Delta t}}\, e^{-\frac{i}{\hbar}\Delta t\, V\left(x + \frac{\eta}{2}, t\right)}\, \psi(x+\eta, t)$$

EXPAND IN $\Delta t$ AND KEEP ONLY LINEAR TERMS IN $\Delta t$ & UP TO QUADRATIC TERMS IN $\eta$

$$\psi(x,t) + \Delta t \frac{\partial \psi}{\partial t}$$

$$= \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta\, e^{-\frac{m}{2\hbar i \Delta t}\eta^2} \left[1 - \frac{i}{\hbar}\Delta t\, V(x,t) + \dots\right] \cdot \left[\psi(x,t) + \eta \frac{\partial \psi}{\partial x} + \frac{1}{2}\eta^2 \frac{\partial^2 \psi}{\partial x^2} + \dots\right]$$

$$\rightsquigarrow \quad \left(\frac{a}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta \, e^{-a\eta^2} = 1 \qquad \left(a = \frac{m}{2\hbar i \Delta t}\right)$$

$$\left(\frac{a}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta \, \eta \, e^{-a\eta^2} = 0$$

$$\left(\frac{a}{\pi}\right)^{1/2} \int_{-\infty}^{+\infty} d\eta \, \eta^2 e^{-a\eta^2} = \left(\frac{a}{\pi}\right)^{1/2} \left(-\frac{1}{2a}\right) \int_{-\infty}^{+\infty} d\left(e^{-a\eta^2}\right) \eta$$

$$= \left(\frac{a}{\pi}\right)^{1/2} \frac{1}{2a} \int_{-\infty}^{+\infty} d\eta \, e^{-a\eta^2}$$

$$= \frac{1}{2a}$$

$$\rightsquigarrow \quad \psi(x,t) + \Delta t \frac{\partial \psi}{\partial t}$$

$$= \psi(x,t) - \frac{i}{\hbar} \Delta t \, V(x,t) \, \psi(x,t)$$

$$+ \left(\frac{\hbar i \Delta t}{m}\right) \cdot \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2}$$

IDENTIFY TERM $O(\Delta t)$

$$\frac{\partial \psi}{\partial t} = -\frac{i}{\hbar} \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x,t)\, \psi \right\}$$

$\Downarrow$

SCHRÖDINGER EQ !

$$\boxed{-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x,t)\, \psi = i\hbar \frac{\partial \psi}{\partial t}}$$

# 2) APPLICATION OF PATH INTEGRALS IN 1D

- FREE PARTICLE $(V=0)$

$$L(x,\dot{x}) = \frac{1}{2} m \dot{x}^2$$

$K_0$ 0 DENOTES FREE PARTICLE

$$\rightsquigarrow K_0(x_f,t_f;x_i,t_i) = \int \tilde{\mathcal{D}} x(t) \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left(\frac{1}{2} m \dot{x}^2\right)}$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar i \Delta t}\right)^{N/2} \cdot \int_{-\infty}^{+\infty} \prod_{k=1}^{N-1} dx_k \; e^{\frac{i}{\hbar}\Delta t \sum_{k=0}^{N-1} \frac{1}{2} m \frac{(x_{k+1}-x_k)^2}{(\Delta t)^2}}$$

$$\text{WITH} \begin{cases} x_0 \equiv x_i \\ x_N \equiv x_f \end{cases}$$

$\rightsquigarrow$ PERFORM FIRST $\int dx_1$

$$\left(\frac{m}{2\pi\hbar i\Delta t}\right)^{2/2} \int dx_1 \; e^{-\left(\frac{m}{2\hbar i \Delta t}\right)\left\{(x_2-x_1)^2 + (x_1-x_0)^2\right\}}$$

$$(x_2-x_1)^2 + (x_1-x_0)^2$$
$$= 2x_1^2 - 2(x_0+x_2)x_1 + x_0^2 + x_2^2$$
$$= 2\left(x_1 - \frac{1}{2}(x_0+x_2)\right)^2 + \frac{1}{2}(x_0-x_2)^2$$

$$= \left(\frac{m}{2\pi\hbar i\Delta t}\right)^{1/2} \cdot \frac{1}{\sqrt{2}} \; e^{-\frac{1}{2}\left(\frac{m}{2\hbar i\Delta t}\right)(x_0-x_2)^2}$$

$\therefore$ AFTER $\int dx_1$ INTEGRATION

$$\left(\frac{m}{2\pi\hbar\, i\, \underline{\underline{2\Delta t}}}\right)^{1/2} e^{-\left(\frac{m}{2\hbar\, i\, \underline{\underline{2\Delta t}}}\right)(x_2 - x_0)^2}$$

$\rightsquigarrow$ MULTIPLY WITH $\left(\frac{m}{2\pi\hbar\, i\Delta t}\right)^{1/2}$ AND $\int dx_2 \; e^{-\frac{m}{2\hbar\, i\Delta t}(x_3 - x_2)^2}$

$$\left(\frac{m}{2\pi\hbar\, i\Delta t}\right)^{1/2} \cdot \left(\frac{m}{2\pi\hbar\, i\, 2\Delta t}\right)^{1/2} \cdot \int dx_2 \; e^{-\frac{m}{2\hbar\, i\Delta t}\left\{(x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2\right\}}$$

$$(x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2$$

$$= \frac{3}{2} x_2^2 - (2x_3 + x_0)\, x_2 + x_3^2 + \frac{1}{2} x_0^2$$

$$= \frac{3}{2}\left(x_2 - \frac{1}{3}(2x_3 + x_0)\right)^2 \underbrace{+ x_3^2 + \frac{1}{2}x_0^2 - \frac{1}{6}\left(4x_3^2 + 4x_0 x_3 + x_0^2\right)}_{\frac{1}{3}(x_3 - x_0)^2}$$

$$= \left(\frac{m}{2\pi\hbar\, i\, \underline{\underline{3\Delta t}}}\right)^{1/2} e^{-\left(\frac{m}{2\hbar\, i\, \underline{\underline{3\Delta t}}}\right)(x_3 - x_0)^2}$$

$\rightsquigarrow$ AFTER $N-1$ INTEGRATIONS.

$$\left(\frac{m}{2\pi\hbar\, i\, N\Delta t}\right)^{1/2} e^{-\left(\frac{m}{2\hbar\, i\, N\Delta t}\right)(\underset{x_N}{\underset{\parallel}{x_f}} - \underset{x_0}{\underset{\parallel}{x_i}})^2}$$

$$\therefore \quad K_0(x_f, t_f\,;\; x_i, t_i)$$

$$= \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar\, i\, N\Delta t}\right)^{1/2} e^{-\frac{m}{2\hbar i N\Delta t}(x_f - x_i)^2}$$

$$\Big\downarrow \qquad N\,\Delta t = t_f - t_i$$

$$\boxed{K_0(x_f, t_f\,;\; x_i, t_i) = \left(\frac{m}{2\pi\hbar\, i\,(t_f - t_i)}\right)^{1/2} \exp\left\{\frac{i m}{2\hbar}\,\frac{(x_f - x_i)^2}{(t_f - t_i)}\right\}}$$

$\rightsquigarrow$ NOTE : FOR CLASSICAL FREE PARTICLE

$$x_{cl}(t) = x_i + \left(\frac{x_f - x_i}{t_f - t_i}\right) t$$

$$\dot{x}_{cl} = \frac{x_f - x_i}{t_f - t_i} \qquad \text{CONSTANT}$$

$$S_{cl} = \int_{t_i}^{t_f} dt \left(\frac{1}{2} m\, \dot{x}_{cl}^2\right)$$

$$\underline{\underline{S_{cl} = \frac{m}{2}\,\frac{(x_f - x_i)^2}{(t_f - t_i)}}}$$

FOR QUANTUM FREE PARTICLE

$$K_0(x_f, t_f\,;\; x_i, t_i) \sim e^{\frac{i}{\hbar} S_{cl}}$$

## MOMENTUM OF FREE PARTICLE

TAKE FREE PARTICLE INITIALLY AT $x_i = 0,\ t_i = 0$

- CLASSICAL $\quad S_{cl} = \frac{m}{2} \frac{x_f^2}{t_f}$

$$\frac{\partial S_{cl}}{\partial x_f} = m \underbrace{\left(\frac{x_f}{t_f}\right)}_{\text{VELOCITY } v} = \underbrace{P}_{\text{MOMENTUM}}$$

- QM : PROBABILITY AMPLITUDE FOR PARTICLE TO BE FOUND AT $x_f, t_f$

$$K_0(x_f, t_f;\ 0, 0) = \left(\frac{m}{2\pi\hbar i t_f}\right)^{1/2} e^{\frac{i}{\hbar}\frac{m}{2}\frac{x_f^2}{t_f}}$$

FOR FIXED $t_f$
↳ OSCILLATES AS $x_f$ VARIES

⇓

PARTICLE BEHAVES AS WAVE

WAVELENGTH : COMPUTED FROM PERIODICITY CONDITION

$$2\pi = \frac{m}{2\hbar t_f}\left[(x_f + \lambda)^2 - x_f^2\right]$$

$\downarrow \quad x_f \gg \lambda$

$$2\pi \simeq \frac{m}{\hbar}\left(\frac{x_f}{t_f}\right)\lambda \implies \boxed{\lambda = \frac{h}{P}}$$

DE BROGLIE

WITH $P = m\frac{x_f}{t_f}$

↑

CLASSICAL MOMENTUM

## ENERGY OF FREE PARTICLE

- CLASSICAL

$$-\frac{\partial S_{cl}}{\partial t_f} = \frac{1}{2} m \frac{x_f^2}{t_f^2} = \frac{1}{2} m v^2 = E$$

- QM

FOR FIXED $x_f$

$K_0$ OSCILLATES AS $t_f$ VARIES

FREQUENCY $\omega = \frac{2\pi}{T}$ $\quad T$ : PERIOD

$$2\pi = \frac{m}{2\hbar} x_f^2 \left[ \frac{1}{t_f} - \frac{1}{t_f + T} \right]$$

$$= \frac{m x_f^2}{2\hbar} \frac{T}{(t_f + T) t_f}$$

$$\downarrow \quad t_f \gg T$$

$$\frac{2\pi}{T} \approx \frac{m}{2\hbar} \frac{x_f^2}{t_f^2} = \frac{E}{\hbar}$$

$$\boxed{E = \hbar \omega}$$

CLASSICAL ENERGY $E = \frac{1}{2} m \frac{x_f^2}{t_f^2}$

# HARMONIC OSCILLATOR

$$L(x,\dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2$$

$$\rightsquigarrow \quad K(x,t;x_i,0) = \int_{x(0)=x_i}^{x(t)=x} \widetilde{\mathcal{D}} x(t') \; e^{\frac{i}{\hbar} S[x(t')]}$$

(CHOOSE $t_i = 0$)

$$S[x(t')] = \int_0^t dt' \left[ \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right]$$

$\rightsquigarrow$ CLASSICAL PATH

$x_{cl}(t)$ IS SOLUTION OF E.L EQ.

$$\frac{\partial L}{\partial x} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

$$-m\omega^2 x - m\ddot{x} = 0$$

$$\ddot{x} = -\omega^2 x$$

$$\downarrow$$

$$x_{cl}(t') = A \cos \omega t' + B \sin \omega t'$$

CONSTRAINTS

$$x_{cl}(0) = x_i \longrightarrow A = x_i$$

$$x_{cl}(t) = x \longrightarrow B = \frac{x - x_i \cos \omega t}{\sin \omega t}$$

$$\boxed{x_{cl}(t') = x_i \cos \omega t' + \frac{(x - x_i \cos \omega t)}{\sin \omega t} \sin \omega t'}$$

⇝ CLASSICAL ACTION

$$S_{cl} \equiv S[x_{cl}] = \frac{m\omega^2}{2}\int_0^t dt' \left\{ \left[-A\sin\omega t' + B\cos\omega t'\right]^2 - \left[A\cos\omega t' + B\sin\omega t'\right]^2 \right\}$$

$$= \frac{m\omega^2}{2}\int_0^t dt' \left\{ +\cos 2\omega t' \, (B^2 - A^2) - \sin 2\omega t' \; 2AB \right\}$$

$$= \frac{m\omega}{4}\left\{ +(B^2 - A^2)\sin 2\omega t' \Big|_0^t + 2AB \cos 2\omega t' \Big|_0^t \right\}$$

$$= \frac{m\omega}{4}\left\{ (-A^2 + B^2)\sin 2\omega t + 2AB(\cos 2\omega t - 1) \right\}$$

$$\boxed{S[x_{cl}] = \frac{m\omega}{2\sin\omega t}\left\{ (x^2 + x_i^2)\cos\omega t - 2 x x_i \right\}}$$

## ⇝ QUANTUM ACTION

$$x(t') = x_{cl}(t') + y(t)$$

↑ DEVIATION FROM CLASSICAL PATH

$$y(0) = 0$$
$$y(t) = 0$$

$$S[x(t')] = \frac{m}{2} \int_0^t dt' \left[ (\dot{x}_{cl} + \dot{y})^2 - \omega^2 (x_{cl} + y)^2 \right]$$

$$= \frac{m}{2} \int_0^t dt' \Big\{ \dot{x}_{cl}^2 - \omega^2 x_{cl}^2 + 2\dot{x}_{cl}\dot{y} - 2\omega^2 x_{cl} y + \dot{y}^2 - \omega^2 y^2 \Big\}$$

NOTE: ↳ $\int_0^t dt' \, 2\dot{x}_{cl}\dot{y} = \cancel{2\dot{x}_{cl} y \Big|_0^t} - \int_0^t dt' \, 2y\ddot{x}_{cl}$

BECAUSE $y(t) = y(0) = 0$

↳ $\int_0^t dt' \left( 2\dot{x}_{cl}\dot{y} - 2\omega^2 x_{cl} y \right)$

$$= -2 \int_0^t dt' \, y \left( \ddot{x}_{cl} + \omega^2 x_{cl} \right)$$

CLASSICAL EQUATION OF MOTION

$$= 0$$

$$\therefore \quad S[x(t')] = S[x_{cl}] + \frac{m}{2}\int_0^t dt' \left[\dot{y}^2 - \omega^2 y^2\right]$$

NOTE THAT $2^{nd}$ TERM DOES NOT DEPEND ON $x$ OR $x_i$

WE CAN THEREFORE WRITE :

$$K(x,t;x_i,0) = K(0,t;0,0)\, e^{\frac{i}{\hbar} S[x_{cl}]}$$

⇝ SHOW THAT

$$K(0,t;0,0) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{1/2}$$

⇝ $$\boxed{K(x,t;x_i,0) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{1/2} e^{\frac{i}{\hbar} S[x_{cl}]}}$$

⇝ NOTE : IN GENERAL → WHEN $S$ CAN BE EXPRESSED THROUGH A QUADRATIC FORM

$$K \sim e^{\frac{i}{\hbar} S[x_{cl}]}$$

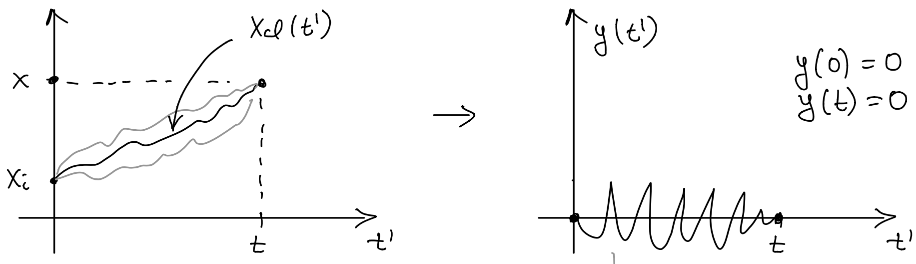
Let's show that

$$K(0,t,0,0) = \left(\frac{m\omega}{2\pi i\hbar \sin(\omega t)}\right)^{1/2}$$

Due to our replacement

deviation from classical path

$$X(t') = X_{cl}(t') + y(t')$$




$$K(0,t,0,0) = \int_0^0 \mathcal{D}y(t') \exp\left(\frac{i}{\hbar}\frac{m}{2}\int_0^t dt'(\dot{y}^2 - \omega^2 y)\right)$$

such path can be written as a Fourier sine series with a fundamental period of $t$

$$y(t') = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi t'}{t}\right)$$

and it is possible to specify a path through the coefficients $a_n$ instead of the function values $y(t_k)$.

The jacobian of this transformation $J$ doesn't depend on $\omega$.

General: all the prefactors that do not depend on $\omega$ we will recover from $\omega = 0$ limit which corresponds to free particle

$$K(0,t,0,0) \xrightarrow{\omega=0} \left(\frac{m}{2\pi\hbar i t}\right)^{1/2}$$

Plug in $y(t')$ into $\exp(\sim)$:

1) $$\frac{m}{2}\int_0^t dt'\,\dot{y}^2 = \frac{m}{2}\int_0^t dt' \sum_{n=1}^{\infty} a_n\left(\frac{n\pi}{t}\right)\cos\left(\frac{n\pi t'}{t}\right)\sum_{m=1}^{\infty} a_m\left(\frac{m\pi}{t}\right)\cos\left(\frac{m\pi t'}{t}\right)$$

$$= \frac{m}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n\pi}{t}\right)\left(\frac{m\pi}{t}\right)\int_0^t dt'\,\cos\left(\frac{n\pi t'}{t}\right)\cos\left(\frac{m\pi t'}{t}\right)$$

using the relation

$\cos(a)\cos(b) = \frac{1}{2}\left(\cos(a-b)+\cos(a+b)\right)$

$$= \frac{m}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n\pi}{t}\right)\left(\frac{m\pi}{t}\right)\frac{t}{2\pi}\left(\underbrace{\frac{\sin(\pi(n-m))}{n-m}}_{\substack{n\neq m \quad 0\\ n=m \quad \pi}} + \underbrace{\frac{\sin(\pi(n+m))}{n+m}}_{\substack{n\neq -m \quad 0\\ \underbrace{n=-m}_{\text{not possible } n,m=[1,\infty)} \quad \pi}}\right)$$

$$= \frac{m}{2}\,\frac{t}{2}\sum_{n=1}^{\infty}\left(\frac{n\pi}{t}\right)^2 a_n^2$$

Similarly

using the relation

$\sin(a)\sin(b) = \frac{1}{2}\left(\cos(a-b)-\cos(a+b)\right)$

2) $$\frac{m\omega^2}{2}\int_0^t dt'\,y^2 = \frac{m\omega^2}{2}\,\frac{t}{2}\sum_{n=1}^{\infty} a_n^2$$

On the assumtion that $[0,t]$ region is divided into discrete steps, there is only a finite number $N$ of coefficients $a_n$

$$K(0,t,0,0) = \underbrace{(J\cdot\ldots)}_{\substack{\text{some factors}\\ \text{that do not}\\ \text{depend on } \omega}} \int_{-\infty}^{+\infty} da_1 \int_{-\infty}^{+\infty} da_2 \ldots \int_{-\infty}^{+\infty} da_N \exp\left\{\frac{im}{2\hbar}\,\frac{t}{2}\sum_{n=1}^{\infty}\left(\left(\frac{\pi n}{t}\right)^2 - \omega^2\right)a_n^2\right\}$$

Since the exp. can be seperated into factors, the integral over each coefficients $a_n$ can be done seperately

$$\int_{-\infty}^{+\infty} da_n \, \exp\left\{ \frac{im}{2\hbar} \frac{t}{2} \left( \frac{\pi^2 n^2}{t^2} - \omega^2 \right) a_n^2 \right\} = (\ldots) \cdot \left( \frac{\pi^2 n^2}{t^2} - \omega^2 \right)^{-1/2} =$$

doesn't depend on $\omega$

Gaussian integral $\int_{-\infty}^{+\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$

$$= (\ldots) \left( 1 - \frac{\omega^2 t^2}{\pi^2 n^2} \right)^{-1/2}$$

Therefore

$$k(0,t,0,0) = (\ldots) \prod_{n=1}^{N} \left( 1 - \frac{\omega^2 t^2}{\pi^2 n^2} \right)^{-1/2} \underset{N \to \infty}{=} (\ldots) \left( \frac{\sin \omega t}{\omega t} \right)^{-1/2}$$

Since

$$k(0,t,0,0) \xrightarrow{\omega = 0} \left( \frac{m}{2\pi\hbar\, it} \right)^{1/2}$$

$$\Rightarrow \boxed{k(0,t,0,0) = \left( \frac{m\omega}{2\pi i \hbar \sin(\omega t)} \right)^{1/2}}$$

#

- PROJECTION OF THE GROUND STATE:

FEYNMAN-KAC FORMULA

$\rightsquigarrow$ $K(x,t;x',0)$

$$= \langle x,t \,|\, x',0 \rangle$$

$$= \langle x \,|\, e^{-\frac{i}{\hbar}\hat{H}t} \,|\, x' \rangle$$

$$= \sum_n \langle x | \psi_n \rangle \langle \psi_n | x' \rangle \, e^{-\frac{i}{\hbar}E_n t}$$

$\rightsquigarrow$ FOR $x = x'$

$$K(x,t;x,0)$$

$$= \sum_n |\psi_n(x)|^2 \, e^{-\frac{i}{\hbar}E_n t}$$

$$\Downarrow$$

$$\int dx \; K(x,t;x,0)$$

$$= \int dx \; \langle x \,|\, e^{-\frac{i}{\hbar}\hat{H}t} \,|\, x \rangle$$

$$= \mathrm{Tr} \; e^{-\frac{i}{\hbar}\hat{H}t}$$

$$= \sum_n \underbrace{\int dx \; |\psi_n(x)|^2}_{1} \; e^{-\frac{i}{\hbar}E_n t}$$

$$\therefore \quad \int dx \; K(x,t;x,0)$$
$$= \sum_n e^{-\frac{i}{\hbar}E_n t} = \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}t} = \sum_n \langle n| e^{-\frac{i}{\hbar}\hat{H}t} |n\rangle$$

ANALYTIC CONTINUATION TO IMAGINARY TIME

$$\boxed{\beta \equiv \frac{it}{\hbar}}$$

$$\boxed{\begin{aligned}&\int dx \; K(x,t;x,0)\\ &= \sum_n e^{-\beta E_n} = \mathrm{Tr}\, e^{-\beta \hat{H}}\end{aligned}}$$

"PARTITION FUNCTION"

FOR $\beta \in \mathbb{R}$ AND POSITIVE

THIS CORRESPONDS TO STATISTICAL MECHANICS PROBLEM

( $\beta$ is $\frac{1}{k_B T}$ WITH T : TEMPERATURE )

IN LIMIT $\beta \to \infty$ ( ZERO TEMPERATURE LIMIT )

ONLY GROUND STATE CONTRIBUTES TO SUM

$$\int dx \; K(x, -i\hbar\beta; x, 0) \xrightarrow[\beta \to \infty]{} e^{-\beta E_0}$$

GROUND STATE ENERGY ( FEYNMAN - KAC FORMULA )

$$\boxed{\begin{aligned}E_0 &= \lim_{\beta\to\infty} \left(-\frac{1}{\beta}\right) \ln \int dx \; K(x, -i\hbar\beta; x, 0)\\ &= \lim_{\beta\to\infty} \left(-\frac{1}{\beta}\right) \ln \mathrm{Tr}\, e^{-\beta\hat{H}}\end{aligned}}$$

⇝ EXAMPLE : HARMONIC OSCILLATOR

↳ $K(x,t\,;\,x',0)$

$$= \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{1/2} e^{\frac{i}{\hbar} S_{cl}}$$

WITH $S_{cl} = \dfrac{m\omega}{2\sin\omega t}\left\{ (x^2+x'^2)\cos\omega t - 2xx' \right\}$

↳ $\int dx \; K(x,t\,;\,x,0)$

$$= \left(\frac{m\omega}{2\pi i\hbar \sin\omega t}\right)^{1/2} \int dx \; e^{-ax^2}$$

WITH $a = -\dfrac{i}{\hbar}\dfrac{m\omega}{\sin\omega t}(\cos\omega t - 1)$

$$= \frac{i}{\hbar}(m\omega)\frac{\sin(\omega t/2)}{\cos(\omega t/2)}$$

$$= \left(\frac{m\omega}{2\pi i\hbar \underbrace{\sin\omega t}_{2\sin\frac{\omega t}{2}\cos\frac{\omega t}{2}}}\right)^{1/2} \left(\frac{\pi\hbar\cos(\omega t/2)}{i m\omega\sin(\omega t/2)}\right)^{1/2}$$

$$= \frac{1}{2i}\,\frac{1}{\sin(\omega t/2)}$$

$$E_0 = \lim_{\beta \to \infty} \left(-\frac{1}{\beta}\right) \ln \int dx \; K(x, -i\hbar\beta; x, 0)$$

$$= \lim_{\beta \to \infty} \left(-\frac{1}{\beta}\right) \ln \left[ \frac{1}{2i} \frac{1}{\sin\left(-i\frac{\hbar\omega}{2}\beta\right)} \right]$$

$$\Big\downarrow \quad \frac{1}{\sin x} = \frac{2i e^{-ix}}{1 - e^{-2xi}}$$

$$= \lim_{\beta \to \infty} \left(-\frac{1}{\beta}\right) \ln \left[ \frac{e^{-\beta\frac{\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}} \right]$$

$$= \lim_{\beta \to \infty} \left(-\frac{1}{\beta}\right) \left\{ -\beta\frac{\hbar\omega}{2} + O\left(e^{-\beta\hbar\omega}\right) \right\}$$

$$\underline{\underline{E_0 = \frac{\hbar\omega}{2}}}$$

↳ IN GENERAL : PARTITION FUNCTION

$$\int dx \; K(x, -i\hbar\beta; x, 0) = \sum_n e^{-\beta E_n}$$

$$\frac{e^{-\beta\frac{\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}} = e^{-\beta\frac{\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-\beta n\hbar\omega}$$

$$\Downarrow$$

$$\underline{\underline{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}}$$

GROUND STATE WAVE FUNCTION

$$\boxed{\begin{aligned} K(x,-i\hbar\beta;x,0) &= \sum_n e^{-\beta E_n} |\psi_n(x)|^2 \\ &\xrightarrow[\beta\to\infty]{} e^{-\beta E_0} |\psi_0(x)|^2 \end{aligned}}$$

<u>H.O</u>: $K(x,-i\hbar\beta;x,0)$

$$= \left( \frac{m\omega}{\pi\hbar\, 2i \sin(-i\hbar\omega\beta)} \right)^{1/2} \exp\left\{ \frac{-i}{\hbar} m\omega x^2 \tan\left(-i\hbar\frac{\omega}{2}\beta\right) \right\}$$

$$\frac{1}{2i\sin(-i\hbar\omega\beta)} = \frac{e^{-\hbar\omega\beta}}{1-e^{-2\hbar\omega\beta}}$$

$$i\tan\left(-i\hbar\frac{\omega}{2}\beta\right) = \frac{1-e^{-\hbar\omega\beta}}{1+e^{-\hbar\omega\beta}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \frac{e^{-\beta E_0}}{\left(1-e^{-\beta 2\hbar\omega}\right)^{1/2}} \exp\left\{ -\frac{m\omega}{\hbar} x^2 \left[ 1 + O\left(e^{-\beta\hbar\omega}\right) \right] \right\}$$

$$\xrightarrow[\beta\to\infty]{} e^{-\beta E_0} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left\{ -\frac{m\omega}{\hbar} x^2 \right\}$$

$$\underline{\underline{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left\{ -\frac{m\omega}{2\hbar} x^2 \right\}}}$$

GROUND STATE W.F. OF H.O.

## 3) EXAMPLE IN 3D : AHARONOV - BOHM EFFECT

- CONSIDER $e^-$ MOVING IN MAGNETIC FIELD

⇝ HAMILTONIAN $\quad H = \frac{1}{2m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)^2$

(MINIMAL SUBSTITUTION)

⇝ LAGRANGE FUNCTION $\quad L = \bar{p}\cdot\dot{\bar{q}} - H$

$\bar{q}$ : COORDINATE , $\dot{\bar{q}}$ VELOCITY

$\bar{p}$ : CONJUGATE MOMENTUM

$$\bar{p} = \frac{\partial L}{\partial \dot{\bar{q}}}$$

$$\dot{\bar{q}} = \frac{\partial H}{\partial \bar{p}} = \frac{1}{m}\left(\bar{p} - \frac{e}{c}\bar{A}\right)$$

$$\Downarrow$$

⇝ CANONICAL MOMENTUM $\quad \underline{\underline{\bar{p} = m\dot{\bar{q}} + \frac{e}{c}\bar{A}}}$

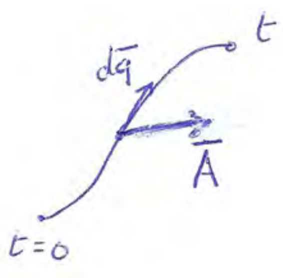
KINETIC MOMENTUM

$$L = \left(m\dot{\bar{q}} + \frac{e}{c}\bar{A}\right)\cdot\dot{\bar{q}} - \frac{m}{2}\dot{\bar{q}}^2$$

$$\underline{\underline{L = \frac{1}{2}m\dot{\bar{q}}^2 + \frac{e}{c}\dot{\bar{q}}\cdot\bar{A}}}$$

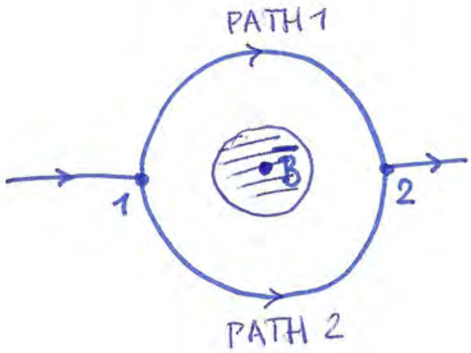
⇝ ACTION $\quad S = \int_0^t dt'\, L(\bar{q}, \dot{\bar{q}})$

$$= \frac{m}{2}\int_0^t dt'\, \frac{d\bar{q}}{dt'}\cdot\frac{d\bar{q}}{dt'} + \frac{e}{c}\int_0^t dt'\,\frac{d\bar{q}}{dt'}\cdot\bar{A}$$




$$= \frac{m}{2}\int_{\text{PATH}} d\bar{q}\cdot\frac{d\bar{q}}{dt'} + \frac{e}{c}\int_{\text{PATH}} d\bar{q}\cdot\bar{A}$$

⟶ CONSIDER CONDUCTING RING
CURRENT ENTERS AT POINT 1, IS EXTRACTED AT POINT 2




CONDUCTING WIRE AROUND SOLENOID → $\bar{B}$ - FIELD

⟶ PROBABILITY AMPLITUDE FOR ELECTRON
TO BE OBSERVED AT POINT 2

$$K(2,1) \sim e^{\frac{i}{\hbar} S(\text{PATH } 1)} + e^{\frac{i}{\hbar} S(\text{PATH } 2)}$$

$$= e^{\frac{i}{\hbar} S(\text{PATH } 1)} \left( 1 + e^{\frac{i}{\hbar} \Delta S} \right)$$

$$\Delta S = S(\text{PATH } 2) - S(\text{PATH } 1)$$

$$= \frac{e}{c} \left\{ \int_{\text{PATH } 2} d\bar{q} \cdot \bar{A} - \int_{\text{PATH } 1} d\bar{q} \cdot \bar{A} \right\}$$

$$= \frac{e}{c} \oint d\bar{q} \cdot \bar{A}$$

<u>NOTE</u> : KINETIC ENERGY TERMS ARE EQUAL
DUE TO SYMMETRY ⟶ AND CANCEL IN DIFFERENCE

STOKES THEOREM

$$\Delta S = \frac{e}{c} \int d\bar{S} . (\bar{\nabla} \times \bar{A})$$

$d\bar{S}$ : NORMAL TO SURFACE OF CURRENT LOOP

$$= \frac{e}{c} \int d\bar{S} . \bar{B}$$

$$\boxed{\Delta S = \frac{e}{c} \Phi}$$

$\Phi$ → MAGNETIC FLUX

$\therefore$ PROB. AMPLITUDE $\sim 1 + e^{\frac{ie}{\hbar c}\Phi}$

PROBABILITY $= |K(2,1)|^2$

$e^{\frac{ie}{\hbar c}\Phi}$ TERM WILL GIVE INTERFERENCE PATTERN WHICH WILL CHANGE BY VARYING $\Phi$ (i.e. $\bar{B}$)

CONSTRUCTIVE INTERFERENCE

$$\frac{e\Phi}{\hbar c} = 2\pi n \implies \boxed{\Phi = \left(\frac{hc}{e}\right) n = n \phi_0}$$

$n \in \mathbb{N}$

$$\boxed{\phi_0 = \frac{hc}{e}}$$ QUANTUM OF FLUX

MAXIMUM CURRENT FOR $\Phi$ EQUAL TO AN INTEGER TIMES $\phi_0$ : AHARONOV - BOHM EFFECT

→ WAS FIRST OBSERVED EXPERIMENTALLY BY CHAMBERS (1960)

# 4) BROWNIAN MOTION AND WIENER PATH INTEGRAL

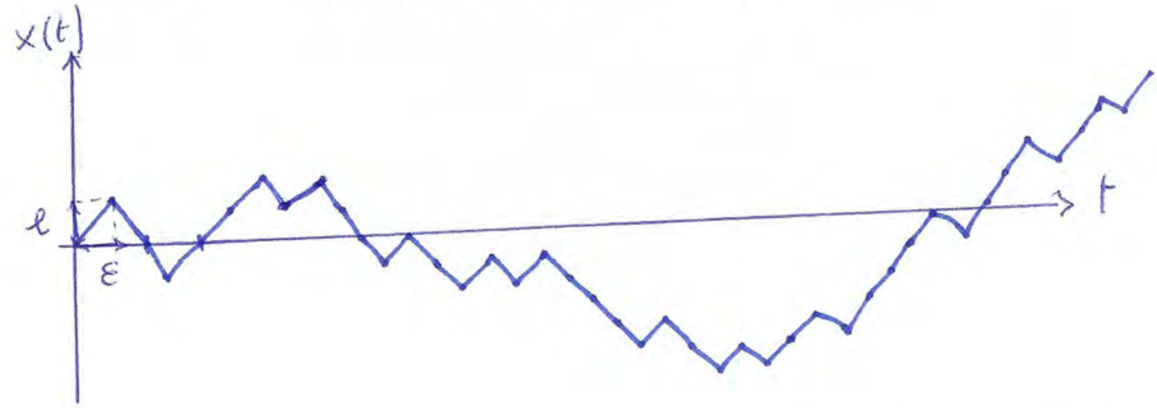
## • RANDOM WALK IN 1 DIMENSION

↳ CONSIDER DISCRETE RANDOM WALK

DISCRETE TIME STEPS $\varepsilon$

DISCRETE SPATIAL STEPS $\ell$ EITHER TO LEFT OR TO RIGHT

SUPPOSE : INITIALLY $t = 0$ , POSITION $x = 0$




↳ PROBABILITY FOR LEFT STEP AND RIGHT STEP ⇒ EACH $\frac{1}{2}$

SUCCESSIVE STEPS ARE STATISTICALLY INDEPENDENT (STOCHASTIC PROCESS)

PROBABILITY FOR TRANSITION FROM $x = j\ell$ TO $x = i\ell$ DURING TIME $\varepsilon$ $(i, j \in \mathbb{Z})$

$$\mathbb{P}(i\ell - j\ell, \varepsilon) = \begin{cases} \frac{1}{2} , & |i-j| = 1 \\ 0 , & \text{OTHERWISE} \end{cases}$$

- HOMOGENEOUS IN SPACE : ONLY DEPENDS ON $i - j$
- ISOTROPIC IN SPACE : SYMMETRIC UNDER $(i, j) \to (-i, -j)$

↳ DISCRETE RANDOM WALK: EXAMPLE OF MARKOV CHAIN

MARKOV CHAIN: CHARACTERIZED BY $\left( \mathbb{P}(t_m), p(0) \right)$

$\mathbb{P}_{ij}(t_m)$: TRANSITION PROBABILITY FROM $j \to i$ AT TIME $t_m$

$p_i(0)$: INITIAL PROBABILITY DISTRIBUTION

$$p_i(t_m) = \sum_j \mathbb{P}_{ij}(t_m)\, p_j(0)$$

NOTE $\quad 0 \leq p_i(0) \leq 1 \qquad \sum_i p_i(0) = 1$

$\quad 0 \leq \mathbb{P}_{ij} \leq 1 \qquad \sum_i \mathbb{P}_{ij} = 1$

TRANSITION PROB. TO STATE $i$ AT TIME $t_m$
DEPENDS ONLY ON STATE $j$ AT TIME $t_{m-1}$
AND NOT ON STATES AT EARLIER TIMES $t_{m-2}, t_{m-3}, \dots$

FOR DISCRETE RANDOM WALK.

$$\mathbb{P}_{ij}(\varepsilon) = \mathbb{P}(i\ell - j\ell, \varepsilon)$$

MARKOV CHAIN: SUCCESSIVE STEPS STATISTICALLY INDEPENDENT

CHOOSE $\quad p_j(0) = \delta_{j0} \quad$, i.e. $x(0) = 0$

↳ $\mathbb{P}$ IN MATRIX NOTATION

MATRIX ELEMENT $ij$

$$\mathbb{P}(\varepsilon) = \frac{1}{2}\left( R(\varepsilon) + L(\varepsilon) \right)$$

$R(\varepsilon)$ : STEP TO RIGHT $\left(R(\varepsilon)\right)_{ij} = \delta_{i,\, j+1}$

$L(\varepsilon)$ : STEP TO LEFT $\left(L(\varepsilon)\right)_{ij} = \delta_{i,\, j-1}$

e.g.

$$\begin{aligned} p_i(\varepsilon) &= \sum_j \mathbb{P}_{ij}(\varepsilon)\; p_j(0) \\ &= \frac{1}{2}\sum_j \left( R_{ij}(\varepsilon) + L_{ij}(\varepsilon) \right) \delta_{j0} \\ &= \frac{1}{2}\left( R_{i0}(\varepsilon) + L_{i0}(\varepsilon) \right) \\ &= \frac{1}{2}\left( \delta_{i1} + \delta_{i-1} \right) \end{aligned}$$

↳ AFTER $n$ TIME STEPS

$$p_i(n\varepsilon) = \sum_j \left(\mathbb{P}^n(\varepsilon)\right)_{ij} \underbrace{p_j(0)}_{\delta_{j0}}$$

NOTE $\mathbb{P} = \frac{1}{2}(R + L)$

$$\mathbb{P}^n = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k} R^k L^{n-k}$$

(BINOMIAL FORMULA)

AS $RL = LR = 1$

$$\downarrow$$

$$\Gamma^{n} = \frac{1}{2^{n}} \sum_{k=0}^{n} \binom{n}{k} R^{2k-n}$$

$$(\Gamma^{n})_{ij} = \frac{1}{2^{n}} \sum_{k=0}^{n} \binom{n}{k} \underbrace{\left(R^{2k-n}\right)_{ij}}_{\delta_{i,\, j+2k-n}}$$

$$\boxed{(\Gamma^{n})_{ij} = \begin{cases} \frac{1}{2^{n}} \binom{n}{\frac{1}{2}(i-j+n)} & \text{, IF } |i-j| \leq n \text{ AND } i-j+n \text{ EVEN} \\ 0 & \text{, OTHERWISE} \end{cases}}$$

$$\Downarrow$$

$$P_i(n\varepsilon) = \sum_j (\Gamma^{n})_{ij} \; P_j(0)$$

$$= (\Gamma^{n})_{i0}$$

$$= \begin{cases} \frac{1}{2^{n}} \binom{n}{\frac{1}{2}(i-n)} & \text{, IF } |i| \leq n \text{ AND } i-n \text{ EVEN} \\ 0 & \text{, OTHERWISE} \end{cases}$$

NOTE

$$\Gamma(i\ell - j\ell, n\varepsilon) = \left(\Gamma^{n}(\varepsilon)\right)_{ij}$$

PROPERTIES:

1) HOMOGENEOUS IN SPACE : DEPENDS ONLY ON $i-\ell$

2) " " TIME : DOES NOT DEPEND ON INITIAL TIME, ONLY ON TIME DIFFERENCE $n$

3) ISOTROPIC IN SPACE $P(-i\ell + j\ell, n\varepsilon) = P(i\ell - j\ell, n\varepsilon)$

↳ RECURSION FORMULA FOR BINOMIAL COEFF

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

PROOF:

$$\text{LEFT} = \frac{(n+1)!}{(n+1-k)!\,k!}$$

$$\text{RIGHT} = \frac{n!}{(n-k)!\,k!} + \frac{n!}{(n-k+1)!\,(k-1)!}$$

$$= \frac{n!}{(n+1-k)!\,k!}\left(n+1-k+k\right) \stackrel{!}{=} \frac{(n+1)!}{(n+1-k)!\,k!}$$

AS $\quad P(il-jl, n\varepsilon) = \frac{1}{2^n}\binom{n}{k}$

FOR $k = \frac{1}{2}(i-j+n)$

IF $|i-j| \leq n$

$i-j+n$ EVEN

$$P(il-jl, (n+1)\varepsilon)$$

$$= \frac{1}{2}\frac{1}{2^n}\binom{n}{k} + \frac{1}{2}\frac{1}{2^n}\binom{n}{k-1}$$

$k = \frac{1}{2}(i-j+n+1)$ $\qquad$ $k-1 = \frac{1}{2}(i-j+n-1)$

$= \frac{1}{2}(\underbrace{i-j+1}+n)$ $\qquad$ $= \frac{1}{2}(\underbrace{i-j-1}+n)$

$$= \frac{1}{2}P((i-j+1)l, n\varepsilon) + \frac{1}{2}P((i-j-1)l, n\varepsilon)$$

OR FOR $x = (i-j)\,\ell$

$t = n\,\varepsilon$

$$\boxed{P(x, t+\varepsilon) = \frac{1}{2} P(x+\ell, t) + \frac{1}{2} P(x-\ell, t)}$$

$$\frac{1}{\varepsilon}\Big( P(x, t+\varepsilon) - P(x,t) \Big)$$

$$= \frac{\ell^2}{2\varepsilon} \cdot \frac{1}{\ell^2} \Big( P(x+\ell, t) - 2P(x,t) + P(x-\ell, t) \Big)$$

$\Downarrow$

MACROSCOPIC DESCRIPTION OF RANDOM WALK

LIMIT $\varepsilon \to 0$

$\ell \to 0$

$\dfrac{\ell^2}{2\varepsilon} = D$ FINITE

$$\boxed{\frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t)}$$ DIFFUSION EQUATION

$D$ : DIFFUSION CONSTANT

## ↳ SOLUTION OF DIFFUSION EQ.

INITIAL CONDITION

$$t=0 \Rightarrow x=0 \quad : \quad P(x,0) = \delta(x)$$

$$\int dx \; P(x,0) = 1$$

$$P(x,t) = \int dk \; e^{ikx} \; \widetilde{P}(k,t)$$

$$\updownarrow$$

$$\widetilde{P}(k,t) = \int \frac{dx}{2\pi} \; e^{-ikx} \; P(x,t)$$

$$\widetilde{P}(k,0) = \frac{1}{2\pi}$$

$$\therefore \quad \frac{\partial}{\partial t} P(x,t) = D \frac{\partial^2}{\partial x^2} P(x,t)$$

$$\downarrow$$

$$\frac{\partial}{\partial t} \widetilde{P}(k,t) = -Dk^2 \; \widetilde{P}(k,t)$$

$$\frac{\partial}{\partial t} \ln \widetilde{P}(k,t) = -Dk^2$$

$$\Downarrow \quad \int_0^t$$

$$\ln \frac{\widetilde{P}(k,t)}{\widetilde{P}(k,0)} = -Dk^2 t$$

$$\boxed{\widetilde{P}(k,t) = \frac{1}{2\pi} \exp\left(-Dk^2 t\right)}$$

$$\rightsquigarrow \quad P(x,t) = \int_{-\infty}^{+\infty} dk \; e^{ikx} \frac{1}{2\pi} e^{-Dk^2 t}$$

$$= \frac{1}{2\pi} \int dk \; e^{-Dt\left(k - \frac{ix}{2Dt}\right)^2 - \frac{x^2}{4Dt}}$$

$$\doteq \frac{1}{2\pi} \sqrt{\frac{\pi}{Dt}} \; e^{-\frac{x^2}{4Dt}}$$

$$\boxed{P(x,t) = \left(\frac{1}{4\pi D t}\right)^{1/2} e^{-\frac{x^2}{4Dt}}}$$

SATISFIES $\int dx \; P(x,t) = 1$

MEAN SQUARE DEVIATION

$$\langle x^2 \rangle_t = \int dx \; x^2 \; P(x,t) \quad , \quad \text{NOTE } \langle x \rangle = 0$$

$$= 2Dt$$

$$\Downarrow$$

$$\sqrt{\langle x^2 \rangle_t} = \sqrt{2Dt}$$

$\hookrightarrow$ FOR ARBITRARY INITIAL CONDITION $x(t_0) = x_0$

$$\boxed{P(x,t;x_0,t_0) = \left(\frac{1}{4\pi D(t-t_0)}\right)^{1/2} e^{-\frac{(x-x_0)^2}{4D(t-t_0)}}}$$

GAUSSIAN

PROBABILITY FOR RANDOM WALKER WHICH INITIALLY STARTS AT $x_0, t_0$ TO BE FOUND AT A LATER TIME $t$ AT $x$

- **IMAGINARY TIME**

↳ DIFFUSION EQ.

$$D \frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial t}$$

$\Downarrow$ ANALYTICALLY CONTINUE TO IMAGINARY TIME i.e $t \Rightarrow it$

$$D \frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial (it)}$$

$$- D \frac{\partial^2 P}{\partial x^2} = i \frac{\partial P}{\partial t}$$

FOR $D = \frac{\hbar}{2m}$ THIS IS SCHRÖDINGER EQ. !

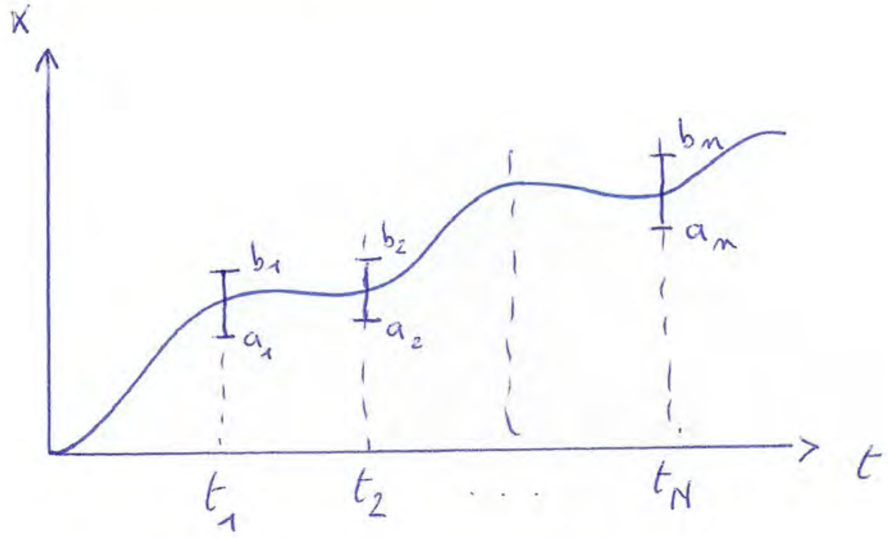
↳
$$\boxed{P(x, it; x_0, it)\Big|_{D=\frac{\hbar}{2m}} = \left(\frac{m}{2\pi \hbar i (t-t_0)}\right)^{1/2} \exp\left\{+\frac{im}{2\hbar}\frac{(x-x_0)^2}{(t-t_0)}\right\}}$$

NOTE 1) THIS IS PRECISELY KERNEL $K(x, t; x_0, t)$ FOR A FREE PARTICLE IN QUANTUM MECHANICS !

2) DIFFUSION EQ : GAUSSIAN $P$ (PROBABILITY)
SCHRÖDINGER EQ : OSCILLATORY KERNEL (PROB. AMPLITUDE!)
$\Downarrow$
→ ADMITS WAVE SOLUTIONS.
→ INTERFERENCE EFFECTS

3) QM FREE PARTICLE SOLUTION MAY BE SEEN AS ANALYTIC CONTINUATION TO IMAG. TIME OF INDETERMINISTIC MOTION OF BROWNIAN PARTICLE

- WIENER PATH INTEGRAL (1921)




↳ CONSIDER BROWNIAN PARTICLE

PROBABILITY TO FIND BROWNIAN PARTICLE
WHICH STARTED AT $t=0$ IN $x=0$

AT TIME $t_1$ IN INTERVAL $[a_1, b_1]$
AT TIME $t_2$ IN INTERVAL $[a_2, b_2]$
⋮
AT TIME $t_N$ IN INTERVAL $[a_N, b_N]$

PRODUCT OF PROBABILITIES DUE TO STOCHASTIC PROCESS

$$\text{PROB}\left\{ x(t_1) \in [a_1, b_1],\ x(t_2) \in [a_2, b_2], \dots, x(t_N) \in [a_N, b_N] \right\}$$

$$= \int_{a_1}^{b_1} dx_1\ \mathbb{P}(x_1, t_1; 0, 0) \int_{a_2}^{b_2} dx_2\ \mathbb{P}(x_2, t_2; x_1, t_1)$$

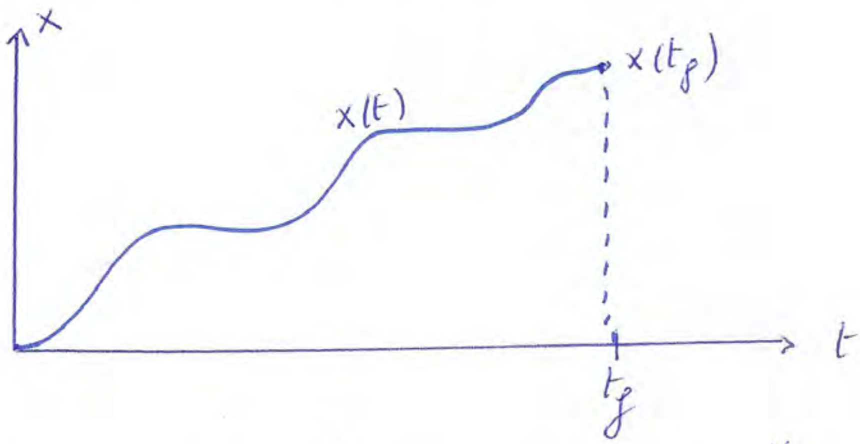
$$\dots \int_{a_N}^{b_N} dx_N\ \mathbb{P}(x_N, t_N; x_{N-1}, t_{N-1})$$

WITH $\mathbb{P}(x_{i+1}, t_{i+1}; x_i, t_i) = \left( \dfrac{1}{4\pi D (t_{i+1} - t_i)} \right)^{1/2} e^{-\dfrac{(x_{i+1} - x_i)^2}{4D(t_{i+1} - t_i)}}$

↳ CONSIDER $t_i - t_{i-1} = \Delta t_i \to 0$ (i.e. $N \to \infty$)

$x_i - x_{i-1} = dx_i \to 0$

PROBABILITY THAT BROWNIAN PARTICLE MOVES ALONG A TRAJECTORY $x(t)$ FROM $x = 0$ AT $t = 0$ TO $x(t_f)$

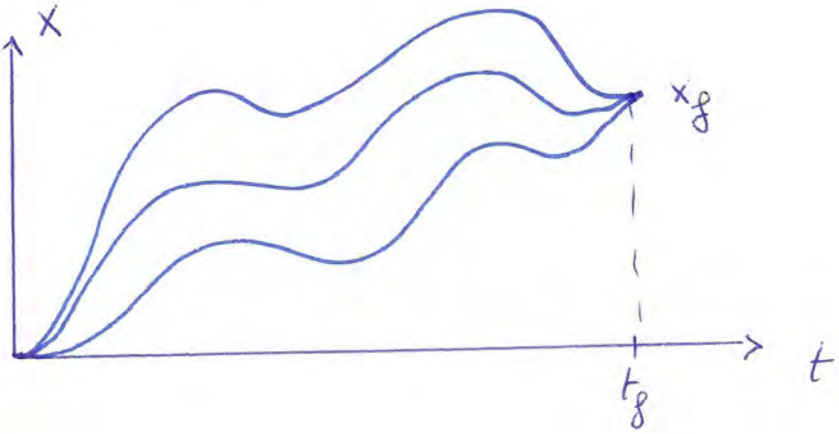



$$\mathbb{P} = \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} \exp\left\{ -\sum_{i=1}^{N} \frac{(x_i - x_{i-1})^2}{4D(t_i - t_{i-1})} \right\} \prod_{i=1}^{N} \left( \frac{1}{4\pi D(t_i - t_{i-1})} \right)^{1/2} dx_i$$

$$= \lim_{\substack{\Delta t_i \to 0 \\ N \to \infty}} \exp\left\{ -\sum_{i=1}^{N} \left( \frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right)^2 \frac{\Delta t_i}{4D} \right\} \prod_{i=1}^{N} \left( \frac{1}{4\pi D \Delta t_i} \right)^{1/2} dx_i$$

$$= \exp\left\{ -\frac{1}{4D} \int_0^{t_f} dt \; \dot{x}^2(t) \right\} \prod_{t=0}^{t_f} \frac{dx(t)}{\sqrt{4\pi D \, dt}}$$

## ↳ PATH INTEGRAL (WIENER)




$P(x_f, t_f; 0, 0)$

CAN BE OBTAINED IN 2 WAYS

1) AS BEFORE

$$P(x_f, t_f; 0, 0) = \left(\frac{1}{4\pi D t_f}\right)^{1/2} e^{-\frac{x_f^2}{4 D t_f}}$$

2) AS SUM OF PROBABILITIES OF ALL PATHS CONNECTING $x(0) = 0$ AND $x(t_f) = x_f$

$$\boxed{P(x_f, t_f; 0, 0) = \int_{C\{0,0;\, x_f t_f\}} d_W\, x(t)}$$

WITH $d_W\, x(t)$ : WIENER MEASURE

$$\boxed{d_W\, x(t) \equiv \exp\left\{-\frac{1}{4D}\int_0^{t_f} dt\; \dot{x}^2(t)\right\} \prod_{t=0}^{t_f} \frac{dx(t)}{\sqrt{4\pi D\, dt}}}$$

WELL DEFINED INTEGRAL

↳ FEYNMAN PATH INTEGRAL (FREE PARTICLE)

$$K(x, t_f; 0, 0) \sim \exp\left\{\frac{i}{\hbar}\int_0^{t_f} dt \, \frac{1}{2} m \dot{x}^2\right\}$$

$\downarrow$

ANALYTICALLY CONTINUE TO
IMAG TIME $\tau = it$

$$K(x, it_f; 0, 0) \sim \exp\left\{\frac{1}{\hbar}\int_0^{it_f} d(it) \, \frac{1}{2} m \dot{x}^2\right.$$

$$\Big\downarrow \; \dot{x}^2 = -\left(\frac{dx}{d\tau}\right)^2$$

$$K(x, \tau; 0, 0) \sim \exp\left\{-\frac{m}{2\hbar}\int_0^{\tau_f} d\tau \left(\frac{dx}{d\tau}\right)^2\right\}$$

$$= \exp\left\{-\frac{1}{4D}\int_0^{\tau_f} d\tau \left(\frac{dx}{d\tau}\right)^2\right\}$$

WITH $D = \frac{\hbar}{2m}$