Examples Sheet 7

Symmetries in Physics

Winter 2019/20

Lecturer: PD Dr. G. von Hippel

1. *Gauge invariance of Wilson lines* (5 P.)

   Consider the theory of a complex scalar field ϕ minimally coupled to Maxwell electromagnetism with action

   $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - V(|\phi|^2).$$

   1. Consider the parallel transporter along a path Γ (a so-called Wilson line)

      $$U_\Gamma = \mathrm{e}^{ie\int_\Gamma dx^\mu A_\mu}$$

      and determine its behaviour under gauge transformation.
   2. Conclude that expressions of the form $\phi^*(y)U_\Gamma\phi(x)$ are gauge invariant if Γ leads from x to y. What happens to expressions of the form $f(U_\Gamma)$ for closed curves Γ?

2. *Goldstone and Higgs effects in scalar QED* (10 P.)

   1. Consider the theory of a complex scalar field ϕ with Lagrangian

      $$\mathcal{L} = \frac{1}{2}|\partial_\mu\phi|^2 - \frac{g^2}{4}\left(|\phi|^2 - v^2\right)^2$$

      and show that it has a continuum of constant solutions $\phi(x) = v\mathrm{e}^{i\alpha}$ with arbitrary constant phase α. Show moreover that by means of the variable transformation

      $$\phi(x) = (v + \sigma(x))\,\mathrm{e}^{i\frac{\pi(x)}{v}}$$

      the Lagrangian density can be rewritten as

      $$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + W(\sigma, \partial_\mu\pi) - V(\sigma)$$

      with

      $$W(\sigma, \partial_\mu\pi) = \frac{2v\sigma + \sigma^2}{2v^2}(\partial_\mu\pi)^2, \quad V(\sigma) = \frac{m^2}{2}\sigma^2 + g^2v\sigma^3 + \frac{g^2}{4}\sigma^4, \quad m^2 = 2g^2v^2.$$

      Which crucial difference exists between the fields σ and π? Interpret this with a view to the shape of the potential $\frac{g^2}{4}\left(|\phi|^2 - v^2\right)^2$.

2. Consider now the theory of a complex scalar field ϕ minimally coupled to the Maxwell field with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}|D_\mu\phi|^2 - \frac{g^2}{4}\left(|\phi|^2 - v^2\right)^2,$$

where $D_\mu = \partial_\mu - iqA_\mu(x)$, and show that the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{\mu^2}{2}A_\mu A^\mu + U(\sigma, A_\mu) - V(\sigma)$$

with

$$U(\sigma, A_\mu) = \frac{2v\sigma + \sigma^2}{2}A_\mu A^\mu, \quad \mu^2 = q^2v^2,$$

by exploiting gauge invariance. Interpret this with respect to the properties of the gauge bosons.

3. *Chiral symmetry of massless QCD* (25 P.)

1. Using the decomposition of the Dirac spinor into a pair of Weyl spinors, show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_{i=1}^{N_f}\overline{\psi}_i\gamma^\mu D_\mu\psi_i$$

for massless QCD with N_f quark flavours is invariant under separate SU(N_f) transformations of the left- and right-handed components $\psi_{L,R} = \frac{1}{2}(1\mp\gamma_5)\psi$ of the Dirac spinor.

2. Write down the Noether currents for the two SU(N_f) symmetries.

3. By forming the sum and the difference of the Noether currents, show that the $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R$ symmetry can also be written as a $\mathrm{SU}(N_f)_V\times\mathrm{SU}(N_f)_A$, with the two factors corresponding to a vector and an axial vector Noether current, respectively.

4. Show that the vector Noether current remains conserved if a degenerate mass term $-m\sum\overline{\psi}_i\psi_i$ is added to the Lagrangian, but that the axial Noether current is no longer conserved.

5. The strong interactions lead to a quark condensate $\langle\mathrm{tr}(Q)\rangle \neq 0$ for the the trace of the matrix $Q_{ij}(x) = \overline{\psi}_i(x)\psi_j(x)$. How does Q transform under $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R$ transformations? Show that the condensate leads to spontaneous breaking of the symmetry associated with the axial vector current, but leaves the vector symmetry intact.

6. Conclude that there must be $(N_f^2 - 1)$ massless Goldstone bosons associated with the spontaneous breaking of the axial symmetry of massless QCD.

7. Using your result for how the mass term breaks the axial symmetry, show that the pseudo-Goldstone bosons associated with the spontaneous breaking of the approximate axial symmetry of QCD with light quarks have masses of order $m_{PS}^2 \propto m$. Which particles are these for $N_f = 2, 3$? What is their parity and why?