

Bahndrehimpulsoperatoren in Kugelkoordinaten

Behauptung:

$$\hat{l}_1 = -\frac{\hbar}{i} \sin(\phi) \frac{\partial}{\partial \theta} - \frac{\hbar}{i} \cot(\theta) \cos(\phi) \frac{\partial}{\partial \phi}, \quad (1)$$

$$\hat{l}_2 = \frac{\hbar}{i} \cos(\phi) \frac{\partial}{\partial \theta} - \frac{\hbar}{i} \cot(\theta) \sin(\phi) \frac{\partial}{\partial \phi}, \quad (2)$$

$$\hat{l}_3 = \frac{\hbar}{i} \frac{\partial}{\partial \phi}, \quad (3)$$

$$\hat{l}^2 = -\hbar^2 \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \right]. \quad (4)$$

Begründung:

1.

$$\begin{aligned} \frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial}{\partial z} \\ &= -r \sin(\theta) \sin(\phi) \frac{\partial}{\partial x} + r \sin(\theta) \cos(\phi) \frac{\partial}{\partial y} \\ &= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}. \end{aligned}$$

\Rightarrow

$$\hat{l}_3 = \frac{\hbar}{i} \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = \frac{\hbar}{i} \frac{\partial}{\partial \phi}.$$

2. Zwischenrechnung

$$\begin{aligned} \hat{l}_1 &= \frac{\hbar}{i} \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) \Rightarrow \frac{\hbar}{i} z \frac{\partial}{\partial y} = -\hat{l}_1 + \frac{\hbar}{i} y \frac{\partial}{\partial z}, \\ \hat{l}_2 &= \frac{\hbar}{i} \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) \Rightarrow \frac{\hbar}{i} z \frac{\partial}{\partial x} = \hat{l}_2 + \frac{\hbar}{i} x \frac{\partial}{\partial z}. \end{aligned}$$

3.

$$\begin{aligned}
\frac{\hbar}{i} \frac{\partial}{\partial \theta} &= \frac{\hbar}{i} \left(\frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial}{\partial z} \right) \\
&= \frac{\hbar}{i} \left(r \cos(\theta) \cos(\phi) \frac{\partial}{\partial x} + r \cos(\theta) \sin(\phi) \frac{\partial}{\partial y} - r \sin(\theta) \frac{\partial}{\partial z} \right) \\
&= \frac{\hbar}{i} \left(\cos(\phi) z \frac{\partial}{\partial x} + \sin(\phi) z \frac{\partial}{\partial y} - r \sin(\theta) \frac{\partial}{\partial z} \right) \\
&= \cos(\phi) \hat{l}_2 - \sin(\phi) \hat{l}_1 \\
&\quad + \frac{\hbar}{i} \underbrace{[x \cos(\phi) + y \sin(\phi) - r \sin(\theta)]}_{r \sin(\theta) \cos^2(\phi) + r \sin(\theta) \sin^2(\phi) - r \sin(\theta)} \frac{\partial}{\partial z} \\
&= r \sin(\theta) \cos^2(\phi) + r \sin(\theta) \sin^2(\phi) - r \sin(\theta) = 0 \\
&= -\sin(\phi) \hat{l}_1 + \cos(\phi) \hat{l}_2. \tag{5}
\end{aligned}$$

4.

$$\begin{aligned}
\frac{\hbar}{i} \cot(\theta) \frac{\partial}{\partial \phi} &= \frac{\hbar}{i} \cot(\theta) \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \\
&= \frac{\hbar}{i} \cot(\theta) \left(r \sin(\theta) \cos(\phi) \frac{\partial}{\partial y} - r \sin(\theta) \sin(\phi) \frac{\partial}{\partial x} \right) \\
&= \frac{\hbar}{i} \left(r \cos(\theta) \cos(\phi) \frac{\partial}{\partial y} - r \cos(\theta) \sin(\phi) \frac{\partial}{\partial x} \right) \\
&= \frac{\hbar}{i} \left(\cos(\phi) z \frac{\partial}{\partial y} - \sin(\phi) z \frac{\partial}{\partial x} \right) \\
&= -\cos(\phi) \hat{l}_1 + \frac{\hbar}{i} \cos(\phi) y \frac{\partial}{\partial z} \\
&\quad - \sin(\phi) \hat{l}_2 - \frac{\hbar}{i} \sin(\phi) x \frac{\partial}{\partial z} \\
&= -\cos(\phi) \hat{l}_1 - \sin(\phi) \hat{l}_2 \\
&\quad + \frac{\hbar}{i} [\cos(\phi) r \sin(\theta) \sin(\phi) - \sin(\phi) r \sin(\theta) \cos(\phi)] \frac{\partial}{\partial z} \\
&= -\cos(\phi) \hat{l}_1 - \sin(\phi) \hat{l}_2. \tag{6}
\end{aligned}$$

5. $-\sin(\phi)$ Gl. (5) $-\cos(\phi)$ Gl. (6) \Rightarrow

$$\hat{l}_1 = -\frac{\hbar}{i} \sin(\phi) \frac{\partial}{\partial \theta} - \frac{\hbar}{i} \cot(\theta) \cos(\phi) \frac{\partial}{\partial \phi}.$$

6. $\cos(\phi)$ Gl. (5) $-\sin(\phi)$ Gl. (6) \Rightarrow

$$\hat{l}_2 = \frac{\hbar}{i} \cos(\phi) \frac{\partial}{\partial \theta} - \frac{\hbar}{i} \cot(\theta) \sin(\phi) \frac{\partial}{\partial \phi}.$$

7. Definition der Leiteroperatoren

$$\hat{l}_\pm := \hat{l}_1 \pm i\hat{l}_2 = \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right). \quad (7)$$

Denn

$$\begin{aligned} \hat{l}_1 \pm i\hat{l}_2 &= -\frac{\hbar}{i} \left[\sin(\phi) \frac{\partial}{\partial\theta} + \cot(\theta) \cos(\phi) \frac{\partial}{\partial\phi} \right] \\ &\quad \pm i \frac{\hbar}{i} \left[\cos(\phi) \frac{\partial}{\partial\theta} - \cot(\theta) \sin(\phi) \frac{\partial}{\partial\phi} \right] \\ &= \hbar [\cos(\phi) \pm i \sin(\phi)] \left[\pm \frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right] \\ &= \hbar e^{\pm i\phi} \left(\pm \frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right). \end{aligned}$$

8.

$$\begin{aligned} \hat{l}^2 &= \hat{l}_1^2 + \hat{l}_2^2 + \hat{l}_3^2 \\ &= \frac{1}{2} \left((\hat{l}_1 + i\hat{l}_2)(\hat{l}_1 - i\hat{l}_2) + (\hat{l}_1 - i\hat{l}_2)(\hat{l}_1 + i\hat{l}_2) \right) + \hat{l}_3^2 \\ &= \frac{1}{2} (\hat{l}_+ \hat{l}_- + \hat{l}_- \hat{l}_+) + \hat{l}_3^2 \\ &= \frac{1}{2} \hbar^2 \left[e^{i\phi} \left(\frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right) e^{-i\phi} \left(-\frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right) \right. \\ &\quad \left. + e^{-i\phi} \left(-\frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right) e^{i\phi} \left(\frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right) \right] \\ &\quad - \hbar^2 \frac{\partial^2}{\partial\phi^2} \\ &= \frac{1}{2} \hbar^2 \left[\left(\frac{\partial}{\partial\theta} + \cot(\theta) + i \cot(\theta) \frac{\partial}{\partial\phi} \right) \left(-\frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right) \right. \\ &\quad \left. + \left(-\frac{\partial}{\partial\theta} - \cot(\theta) + i \cot(\theta) \frac{\partial}{\partial\phi} \right) \left(\frac{\partial}{\partial\theta} + i \cot(\theta) \frac{\partial}{\partial\phi} \right) \right] \\ &\quad - \hbar^2 \frac{\partial^2}{\partial\phi^2} \\ &= \frac{1}{2} \hbar^2 \left[-2 \frac{\partial^2}{\partial\theta^2} - 2 \cot(\theta) \frac{\partial}{\partial\theta} - 2 \cot^2(\theta) \frac{\partial^2}{\partial\phi^2} \right] - \hbar^2 \frac{\partial^2}{\partial\phi^2} \\ &= -\hbar^2 \left[\frac{\partial^2}{\partial\theta^2} + \cot(\theta) \frac{\partial}{\partial\theta} + \cot^2(\theta) \frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\phi^2} \right] \\ &= -\hbar^2 \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta} \left(\sin(\theta) \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial\phi^2} \right]. \end{aligned}$$