Practice exam
Theoretical Physics 5 : WS 2019/2020
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Question 1. (25 points): Spontaneous symmetry breaking

Consider the following Lagrangian for a complex scalar field $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$:

$$\mathcal{L} \;=\; (\partial^\mu \phi^*)(\partial_\mu \phi) + \mu^2 \phi^* \phi - \frac{\lambda}{2}\,(\phi^* \phi)^2.$$

Clearly, for $\mu^2 = -m_0^2 c^2/\hbar^2$ and $\lambda = 0$, one recovers the familiar Klein-Gordon Lagrangian. Moreover, this Lagrangian is also invariant under the phase transformation $\phi \to e^{i\alpha}\phi$.

(a) (5 points) Treating ϕ and ϕ^* as independent, show that the corresponding Hamiltonian is given by

$$H \;=\; \int d^3x \left(\frac{1}{c^2}\,\dot{\phi}^*\dot{\phi} + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi - \mu^2 \phi^*\phi + \frac{\lambda}{2}\,(\phi^*\phi)^2 \right).$$

Hint: Remember that the conjugate momentum π of a field ϕ is given by $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ and the Hamiltonian by $H = \int d^3x \left(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}\right)$.

(b) (10 points) Find the condition on classical field configurations $\phi_0(x)$ which assures minimization of the energy. Show that there is an infinite set of such configurations and that they are all related by the phase transformation.
Hint: Try first to minimize the "kinetic" term $\frac{1}{c^2}\,\dot{\phi}^*\dot{\phi} + \vec{\nabla}\phi^* \cdot \vec{\nabla}\phi$, then minimize the "potential" term $-\mu^2\phi^*\phi + \frac{\lambda}{2}\,(\phi^*\phi)^2$.

(c) (10 points) Suppose that the system is near the minimum $\phi_0 = \mu/\sqrt{\lambda}$. Then it is convenient to define

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}\,[\sigma(x) + i\theta(x)]\,,$$

where the fields $\sigma(x)$ and $\theta(x)$ describe (real) fluctuations around the minimum. Rewrite $\mathcal{L}$ in terms of these fluctuations. What are the masses of the σ and θ fields?
Hint: Remember that the mass can be read off from the coefficient quadratic in the field, *e.g.* for a real scalar field Φ with mass m_0 we had the quadratic term $-\frac{1}{2}\,\frac{m_0^2 c^2}{\hbar^2}\,\Phi^2$.

Question 2. (15 points): Dirac particle in a square-well potential

Consider a Dirac particle travelling along the positive z-direction and subject to the square-well potential

$$V(z) = \begin{cases} 0 & \text{when } z < -a/2 & \text{(region I)} \\ V_0 & \text{when } -a/2 \le z \le a/2 & \text{(region II)} \\ 0 & \text{when } a/2 < z & \text{(region III),} \end{cases}$$

where the width $a > 0$ and the depth $V_0 < 0$. In regions I and III, the eigenvalue Dirac equation takes the form

$$\left(\vec{\alpha} \cdot \hat{\vec{p}}\, c + \beta m_0 c^2\right) \psi \;=\; E\psi,$$

while in region II, it has the form

$$\left(\vec{\alpha} \cdot \hat{\vec{p}}\, c + \beta m_0 c^2\right) \psi \;=\; (E - V_0)\psi.$$

Hints: Remember that the matrices $\vec{\alpha}$ and β in the standard (Dirac) representation are given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

(a) (10 points) Write down the general solution $\psi(z)$ in the three regions with the spin in the z-direction.

(b) (5 points) Impose the continuity condition at $z = \pm a/2$ and write the corresponding equations.

Question 3. (15 points): Bose-Hubbard model

The Bose-Hubbard model gives an approximate description of the physics of interacting bosons on a lattice. It can be used to study systems such as bosonic atoms on an optical lattice, *i.e.* a periodic trap formed by the interference of counterpropagating laser beams. This system resembles a crystal in the sense that the atoms are in a periodic potential. The Hamiltonian of this model is given by (latin indices refer to lattice sites)

$$H = -t \sum_{<i,j>} \left(C_i^\dagger C_j + C_j^\dagger C_i \right) + \frac{U}{2} \sum_i C_i^\dagger C_i \left(C_i^\dagger C_i - 1 \right),$$

where $< i, j >$ - the sum is restricted over first neighbors only, and $U > 0$.

(a) (5 points) Provide an interpretation of each term of this Hamiltonian.

(b) (10 points) Show that in this model, the number of particles is conserved.

Question 4. (45 points): $\pi^+ e^-$ elastic scattering

Consider elastic unpolarized $\pi^+ e^-$ scattering in Quantum Electrodynamics: $\pi^+(p)\, e^-(k) \to \pi^+(p')\, e^-(k')$. Treat the electron as a massless Dirac particle and the pion as a massive Klein-Gordon particle with mass M.

(a) (5 points) Write down the expression for the differential cross-section in the laboratory frame in terms of the matrix element.

(b) (15 points) Integrate over the pion phase space and electron momentum and obtain the expression for angular differential cross section in terms of the electron scattering angle and the matrix element.

(c) (15 points) Calculate the squared matrix element $|M|^2$ (obtained as an average over initial spin configurations and sum over final electron spin configurations) in terms of electron and pion momenta. Show that the leptonic tensor can be expressed as a trace of γ-matrices.

(d) (10 points) Calculate $|M|^2$ in the laboratory frame and combine all elements to obtain the differential cross section.

Question 5. (20 points): Weyl representation

In the standard Dirac representation, Dirac matrices have the form

$$\gamma_s^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma}_s = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

while in the so-called *Weyl representation*, they have the form

$$\gamma_W^\mu \;=\; \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

with $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$.

(a) (5 points) Write down a unitary matrix S connecting both representations $\gamma_s^\mu = S\gamma_W^\mu S^{-1}$.

(b) (5 points) Write the γ_5 matrix in both representations.

(c) (10 points) In the Weyl representation with $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, show that the bispinors ψ_L and ψ_R are independent for massless particles, and write down the eigenstates of the chirality operator γ_5 with their corresponding eigenvalues.

Formulary:

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Dirac matrices:

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\rho\nu} - g^{\mu\rho}g^{\nu\sigma}\right)$$