Examples Sheet 6

Symmetries in Physics

Winter 2019/20

Lecturer: PD Dr. G. von Hippel

Vacation sheet – Return 09.01.2020 – with up to 20 bonus points.

1. $\mathrm{SL}(2,\mathbb{C})$ *as the double cover of the Lorentz group* (15 P.)

   1. Let $\sigma_\mu = (\mathbb{1}, \boldsymbol{\sigma})$, where σ_i are the Pauli matrices. For a four-vector v define the 2×2 matrix $V(v)$ by
      $$V(v) = v_\mu \sigma^\mu .$$
      Show that $\det V(v) = v^2$.
   2. Show that $v_\mu = \frac{1}{2}\,\mathrm{tr}\,[\bar{\sigma}_\mu V(v)]$, where $\bar{\sigma}_\mu = (\mathbb{1}, -\boldsymbol{\sigma})$.
   3. Show that $A \in \mathrm{SL}(2,\mathbb{C})$ induces a Lorentz transformation $v \mapsto v'$ through
      $$V(v') = AV(v)A^\dagger .$$
   4. Deduce that
      $$\Lambda^\mu_\nu(A) = \frac{1}{2}\,\mathrm{tr}\left[\bar{\sigma}^\mu A \sigma_\nu A^\dagger\right]$$
      defines a homomorphism from $\mathrm{SL}(2,\mathbb{C})$ into the proper orthochronous Lorentz group.
   5. Identify the kernel of the homomorphism and conclude that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of the proper orthochronous Lorentz group.

2. *Weyl spinors* (10 P.)

    1. Derive the generators of boosts and rotations in the $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ representations of the Lorentz group.

    2. Conclude that these representations of the Lorentz group can be written as

    $$D^{(\frac{1}{2},0)}(\boldsymbol{\xi}) = \mathrm{e}^{\frac{i}{2}\boldsymbol{\xi}\cdot\boldsymbol{\sigma}}$$

    and

    $$D^{(0,\frac{1}{2})}(\boldsymbol{\xi}) = \mathrm{e}^{\frac{i}{2}\boldsymbol{\xi}^{*}\cdot\boldsymbol{\sigma}}$$

    where $\xi_i = \theta_i - i\zeta_i$.

3. *The Dirac algebra* (15 P.)

    1. Let γ_μ be the generators of the Dirac (Clifford) algebra defined by the anticommutation relation

    $$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu},$$

    and let

    $$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu].$$

    Show that a representation ρ of the Dirac algebra gives a representation $\tilde{\rho}$ of the Lie algebra of the Lorentz group via $\tilde{\rho}(L_{\mu\nu}) = \frac{1}{2}\rho(\sigma_{\mu\nu})$.

    2. Show that a representation (the Weyl representation) of the Dirac algebra is given by

    $$\rho(\gamma_0) = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \rho(\gamma_i) = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}.$$

    3. Give the generators of boosts and rotations in the Weyl representation; conclude that the corresponding representation of the Lorentz group is reducible, and give its decomposition into irreps.

    4. Show that another representation (the Pauli representation) of the Dirac algebra is given by

    $$\rho(\gamma_0) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \rho(\gamma_i) = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}.$$

4. * *Algebraic solution of the hydrogen atom* (+20 P.)

In the lectures, we discussed the dynamical SO(4) symmetry of the Kepler-Coulomb potential, leaving most explicit calculations in the quantum mechanical case as exercises. In this question, you are asked to fill in the missing pieces. *By solving this "starred" question, you can earn up to 20 bonus points.*

1. * Show that the components of the Runge-Lentz-Pauli operator satisfy

$$[\hat{H}, \hat{M}_i] = 0$$

2. * Show that the components of the Runge-Lentz-Pauli operator satisfy

$$[\hat{M}_i, \hat{M}_j] = i\epsilon_{ijk} \left(-\frac{2\hat{H}}{m} \right) \hat{L}_k$$

3. * Show that the Runge-Lentz-Pauli operator and orbital angular momentum operator satisfy

$$\hat{\boldsymbol{M}} \cdot \hat{\boldsymbol{L}} = \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{M}} = 0$$

4. * Show that the Runge-Lentz-Pauli operator squares to

$$\hat{\boldsymbol{M}}^2 = \frac{2\hat{H}}{m} \left(\hat{\boldsymbol{L}}^2 + \hbar^2 \hat{\mathbb{1}} \right) + e^4$$

where the prefactor $\hbar^2 = 1$ can be restored on dimensional grounds.

Merry Christmas and a Happy New Year!