2 problems. Total number of points given is 25.
Grading: 20+ excellent, 15+ good, below 10 is incomplete.
Deadline: Thursday, December 19.

---

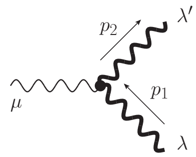
1. **Spin-1 QED from Yang-Mills theory**

   Consider the following Lagrangian

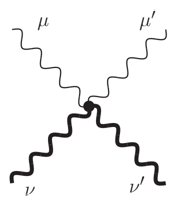
$$\mathcal{L} = -\frac{1}{2} G^*_{\mu\nu} G^{\mu\nu} + M_W^2 \, W^*_\mu W^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}$$

   that describes the photon field A_μ by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the massive charged complex vector field W_μ by $G_{\mu\nu} = D_\mu W_\nu - D_\nu W_\mu$, where $D_\mu = \partial_\mu + ieA_\mu$.

   (a) {2 pts} Convince yourself that the Feynman rules interaction vertices are these:



$$= \quad -ie \left[(p_1^\mu + p_2^\mu) g^{\lambda\lambda'} - p_1^{\lambda'} g^{\mu\lambda} - p_2^{\lambda} g^{\mu\lambda'} \right] \tag{2}$$



$$= \quad -ie^2 \left(2 g^{\mu\mu'} g^{\nu\nu'} - g^{\mu\nu} g^{\mu'\nu'} - g^{\mu\nu'} g^{\mu'\nu} \right) \tag{3}$$

   where the thicker lines depict vector boson W and the thinner ones are the photons.

   (b) {5 pts} Taking the propagator of the vector field in the form

$$\Delta_W^{\mu\nu}(k) = -i \frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M_W^2}}{k^2 - M_W^2 + i\epsilon},$$

   obtain the amplitude of the tree-level **forward** ($t = 0$) Compton scattering process in this theory using FORM. Make sure that the answer can be reduced to

$$e^2 \left[2 \, \epsilon_q \cdot \epsilon_q^* \, \chi_p \cdot \chi_p^* + \frac{\nu}{2M_W} \left(\epsilon_q \cdot \chi_p \, \epsilon_q^* \cdot \chi_p^* - \epsilon_q \cdot \chi_p^* \, \chi_p \cdot \epsilon_q^* \right) \right]$$

where ϵ_{k_i} are the photon polarization vectors and χ_{k_i} are the polarization vectors of the charged vector boson, $\nu = p \cdot q / M_W$.

Now turn to the SU(2) massive Yang-Mills theory with Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathcal{F}^a_{\mu\nu}\mathcal{F}^{\mu\nu}_a + \frac{M_W^2}{2}\left[\,\left(\mathcal{A}^1_\mu\right)^2 + \left(\mathcal{A}^2_\mu\right)^2\right], \quad a = 1,\,2,\,3; \tag{4}$$

where $\mathcal{F}^{\mu\nu}_a = \partial^\mu \mathcal{A}^\nu_a - \partial^\nu \mathcal{A}^\mu_a - e f^{abc}\mathcal{A}^\mu_b \mathcal{A}^\nu_c$, $f^{abc} = \epsilon^{abc}$ are the structure constants of SU(2) algebra. Let the 3$^{\text{rd}}$ (massless) component describe the photon $\mathcal{A}^\mu_3 = A^\mu$, and the two other ones describe the charged vector boson $W^\pm$ with the following complex field: $W_\mu = \frac{1}{\sqrt{2}}(\mathcal{A}^1_\mu + i\mathcal{A}^2_\mu)$ and $W^*_\mu = \frac{1}{\sqrt{2}}(\mathcal{A}^1_\mu - i\mathcal{A}^2_\mu)$.

(c) {5 pts} Obtain the amplitude of the tree-level **forward** Compton scattering process $\gamma W^\pm \to \gamma W^\pm$ in this theory using FORM. Compare results with (b).

2. **Vacuum polarization in the Lamb shift**

Consider the contribution of the QED vacuum polarization (VP) to the hydrogen Lamb shift (see Fig.1).

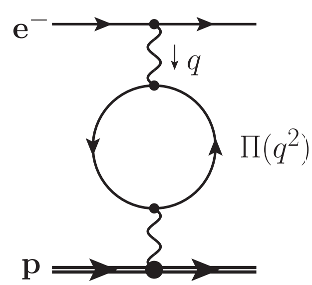


Figure 1: Vacuum polarization contribution to the Lamb shift at the first order.

The VP has the well-known form:

$$\Pi^{\mu\nu}(q) = (g^{\mu\nu}q^2 - q^\mu q^\nu)\,\Pi(q^2), \tag{5}$$

where we use the flat Minkowski metric, with diag $g^{\mu\nu} = (1, -1, -1, -1)$. The renormalized $\Pi(q^2)$ satisfies the once-subtracted dispersion relation:

$$\Pi(q^2) = \frac{1}{\pi} \int_{t_0}^{\infty} dt \, \frac{\text{Im } \Pi(t)}{t - q^2 - i0^+} \frac{q^2}{t}, \tag{6}$$

where $t_0 = 4m^2$ is the lowest particle-production threshold (m is the mass of the lepton in the loop).

The corresponding correction to the Coulomb potential $V_C(r) = -\alpha/r$ is given by:

$$\delta V(r) = \int \frac{d\vec{q}}{(2\pi)^3} \, e^{i\vec{q}\cdot\vec{r}} \, \delta V(\vec{q}^{\,2}), \qquad \delta V(\vec{q}^{\,2}) = -\frac{4\pi\alpha}{\vec{q}^{\,2}} \, \Pi(-\vec{q}^{\,2}) \, . \tag{7}$$

where the retardation effects can be neglected (i.e., $q_0 = 0$) because they are of higher order in α. This correction is referred to as the Uehling potential.

(a) {2 pts} Using the above dispersion relation, show that the Uehling potential is given by the Yukawa potential with a dispersed mass.

(b) {4 pts} Using the hydrogen wave-functions $\phi_{nlm}(\vec{r})$, calculate the first-order $2P - 2S$ Lamb shift due to the Yukawa potential with mass M, i.e.:

$$\int d\vec{r} \left(|\phi_{210}|^2 - |\phi_{200}|^2 \right) \frac{e^{-Mr}}{r} = -\frac{\alpha^3 m_r^3 M^2}{2(M + \alpha m_r)^4},$$

where m_r is the reduced mass of hydrogen.

(c) {2 pts} Combining the above two results, write down the general first-order VP contribution to the Lamb shift

$$\Delta E(2P - 2S) = \int d\vec{r} \left(|\phi_{210}|^2 - |\phi_{200}|^2 \right) \delta V(r)$$

in terms of $\text{Im } \Pi(t)$.

(d) {5 pts} Substitute the one-loop result for the electron VP:

$$\text{Im } \Pi^{(1)}(t) = -\frac{\alpha}{3} \left(1 + \frac{2m_e^2}{t} \right) \sqrt{1 - \frac{4m_e^2}{t}},$$

and, using Mathematica, obtain the numerical value (in eV) for the Lamb shift in normal and muonic hydrogen.