Exercise sheet 9
Theoretical Physics 5 : WS 2019/2020
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (20 points) Ground state Dirac Coulomb wave functions

Consider the $1s_{1/2}$ ground state of an electron in a hydrogen-like ion with nuclear charge Z. Derive the **normalized** radial Dirac Coulomb wave functions $F(\rho)$ and $G(\rho)$ from the power series expansion method considered at the lecture.
Hints: Recall

$$F(\rho) = \sqrt{k_2} e^{-\rho/2} \sum_{m=0}^{n'} a_m \rho^{m+\gamma}$$

$$G(\rho) = \sqrt{k_1} e^{-\rho/2} \sum_{m=0}^{n'} b_m \rho^{m+\gamma},$$

where $n' = n - (j + 1/2)$, $\rho = 2\sqrt{k_1 k_2} r$ and $k_{1,2} = \frac{1}{\hbar c}(\pm E + m_0 c^2)$. As was shown $\gamma = \sqrt{(j + \frac{1}{2})^2 - (Z\alpha)^2}$ and the spectrum is

$$E_{nj} = \frac{m_0 c^2}{\sqrt{1 + \left(\frac{Z\alpha}{n-(j+1/2)+\gamma}\right)^2}}.$$

The $1s_{1/2}$ state corresponds to $n = 1$ and $j = 1/2$ so that F and G consist of a single term. The normalization condition implies

$$\int_0^\infty \mathrm{d}r \, \left[F^2(r) + G^2(r)\right] = 1.$$

Use that the gamma function is defined via

$$\Gamma(z) = \int_0^\infty \mathrm{d}x \, x^{z-1} e^{-x}.$$

Exercise 2. (30 points) : The angular momentum operator

a) (*15 p.*) Starting from the transformation law for the classical Dirac field under Lorentz transformations show that the generators of these transformations are given by

$$M_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}$$

b) (*15 p.*) The angular momentum of the Dirac field is

$$M_{\mu\nu} = \int \mathrm{d}^3\vec{x}\, \psi^\dagger(x) \left[i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2}\sigma_{\mu\nu} \right] \psi(x).$$

Prove that

$$[M_{\mu\nu}, \psi(x)] = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)\psi(x) - \frac{1}{2}\sigma_{\mu\nu}\psi(x).$$

Exercise 3. (30 points) : Dirac field

Using the normal mode expansion of the Dirac field

$$\begin{aligned}
\psi(\vec{x}, t) &= \sum_{\vec{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V} \right)^{1/2} \left[b(\vec{p}, s_z)\, u(\vec{p}, s_z) e^{-\frac{i}{\hbar} p\cdot x} + d^\dagger(\vec{p}, s_z)\, v(\vec{p}, s_z) e^{\frac{i}{\hbar} p\cdot x} \right] \\
\bar{\psi}(\vec{x}, t) &= \sum_{\vec{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V} \right)^{1/2} \left[b^\dagger(\vec{p}, s_z)\, \bar{u}(\vec{p}, s_z) e^{\frac{i}{\hbar} p\cdot x} + d(\vec{p}, s_z)\, \bar{v}(\vec{p}, s_z) e^{-\frac{i}{\hbar} p\cdot x} \right]
\end{aligned}$$

and the equal-time creation and annihilation operators anticommutation relations

$$\begin{gathered}
\{b(\vec{p}, s_z), b^\dagger(\vec{p}\,', s_z')\} = \delta_{\vec{p},\vec{p}\,'} \delta_{s_z, s_z'}, \\
\{d(\vec{p}, s_z), d^\dagger(\vec{p}\,', s_z')\} = \delta_{\vec{p},\vec{p}\,'} \delta_{s_z, s_z'}, \\
\{b(\vec{p}, s_z), b(\vec{p}\,', s_z')\} = 0, \quad \{d(\vec{p}, s_z), d(\vec{p}\,', s_z')\} = 0, \\
\{b(\vec{p}, s_z), d(\vec{p}\,', s_z')\} = 0, \quad \{d(\vec{p}, s_z), b(\vec{p}\,', s_z')\} = 0, \\
\{b(\vec{p}, s_z), d^\dagger(\vec{p}\,', s_z')\} = 0, \quad \{d(\vec{p}, s_z), b^\dagger(\vec{p}\,', s_z')\} = 0;
\end{gathered}$$

a) (*5 p.*) Show that the following equal time anticommutation relations hold:

$$\begin{aligned}
\{\psi_\alpha(\vec{x}, t), \psi_\beta(\vec{x}\,', t)\} &= 0, \\
\left\{\psi_\alpha^\dagger(\vec{x}, t), \psi_\beta^\dagger(\vec{x}\,', t)\right\} &= 0;
\end{aligned}$$

b) (*15 p.*) Express $H = \int \mathrm{d}^3\vec{x}\, N\left(\bar{\psi}(-i\hbar\gamma^i\partial_i + m_0c^2)\psi\right)$ in terms of creation and annihilation operators.

c) (*10 p.*) Do the same for the momentum $\vec{P} = -i\hbar \int \mathrm{d}^3\vec{x}\, N\left(\psi^\dagger \vec{\nabla}\psi\right)$.

Exercise 4. (20 points)

Calculate $[H, b^\dagger(\vec{p}, s_z) b(\vec{p}, s_z)]$.