Exercise sheet 8
Theoretical Physics 5 : WS 2019/2020
Lecturers : Prof. M. Vanderhaeghen, Dr. I. Danilkin

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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (25 points) : Projection operators

a) (*15 p.*) Show that $\Sigma_\pm \equiv \Sigma(\pm n) = \frac{1}{2}(1 \pm \gamma_5 \not{n})$ are two complete sets of projection operators, *i.e.* satisfy the conditions

$$\Sigma_i \Sigma_j = \delta_{i,j}\Sigma_i \qquad \text{and} \qquad \sum_i \Sigma_i = 1, \qquad \text{where} \qquad i,j = \pm.$$

Show that $\Sigma_{+,-}$ are the projection operators on positive- and negative-spin projection solutions for arbitrary direction of normalised spin vector ($n_\mu n^\mu = -1$).

b) (*10 p.*) Do operators $\Sigma_\pm$ and $\Lambda_\pm = \frac{1}{2m_0c}(\pm\not{p} + m_0 c)$ have a common system of eigenfunctions? Why?

Exercise 2. (60 points) :
Normalization and completeness of Dirac spinors

Recall that the eigenfunctions of the Dirac equation have the form $\psi_r(x) = \omega_r(\vec{p})e^{-i\lambda_r p_\mu x^\mu/\hbar}$, with $r = 1\ldots4$, where $\lambda_{1,2} = 1$, $\lambda_{3,4} = -1$ and

$$\omega_1(\vec{p}) = N\sqrt{\frac{E+m_0c^2}{2m_0c^2}}\begin{pmatrix} 1 \\ 0 \\ \frac{\vec{\sigma}\cdot\vec{p}\,c}{E+m_0c^2}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}, \qquad \omega_2(\vec{p}) = N\sqrt{\frac{E+m_0c^2}{2m_0c^2}}\begin{pmatrix} 0 \\ 1 \\ \frac{\vec{\sigma}\cdot\vec{p}\,c}{E+m_0c^2}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix},$$

$$\omega_3(\vec{p}) = N\sqrt{\frac{E+m_0c^2}{2m_0c^2}}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}\,c}{E+m_0c^2}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{pmatrix}, \qquad \omega_4(\vec{p}) = N\sqrt{\frac{E+m_0c^2}{2m_0c^2}}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}\,c}{E+m_0c^2}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{pmatrix}.$$

For convenience, we introduce a common notation for positive-energy spinors:

$$u(\vec{p},s_z) = N\sqrt{\frac{E+m_0c^2}{2m_0c^2}}\begin{pmatrix} \chi_{s_z} \\ \frac{\vec{\sigma}\cdot\vec{p}\,c}{E+m_0c^2}\chi_{s_z}\end{pmatrix} \qquad \text{with} \qquad \chi_{s_z=\frac{1}{2}} = \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad \chi_{s_z=-\frac{1}{2}} = \begin{pmatrix}0\\1\end{pmatrix},$$

and for negative-energy ones:

$$v(\vec{p}, s_z) = N\sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}\,c}{E+m_0c^2}\chi'_{s_z} \\ \chi'_{s_z} \end{pmatrix} \qquad \text{with} \qquad \chi'_{s_z=\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \chi'_{s_z=-\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In the following, the normalization of spinors is chosen to be $N = 1$, so that $\bar{u}(p, s_z)u(p, s'_z) = \delta_{s_z s'_z}$ and $\bar{v}(p, s_z)v(p, s'_z) = -\delta_{s_z s'_z}$.

Prove the following relations :

a) (*15 p.*) the normalization condition for spinors w

$$\bar{w}_r(\vec{p})w_{r'}(\vec{p}) = w_r^\dagger(\vec{p})\gamma^0 w_{r'}(\vec{p}) = \delta_{rr'}\lambda_r$$

b) (*15 p.*) the completeness of spinors w

$$\sum_{r=1}^{4} \lambda_r (w_r(\vec{p}))_\alpha (\bar{w}_r(\vec{p}))_\beta = \delta_{\alpha\beta}, \quad \alpha, \beta = 1...4$$

c) (*15 p.*)

$$u^\dagger(p, s_z)u(p, s'_z) = \frac{E_p}{m_0c^2}\,\delta_{s_z s'_z},$$
$$v^\dagger(p, s_z)v(p, s'_z) = \frac{E_p}{m_0c^2}\,\delta_{s_z s'_z},$$

d) (*15 p.*)

$$\sum_{s_z} u(p, s_z)\bar{u}(p, s_z) = \frac{\not{p} + m_0c}{2m_0c},$$
$$\sum_{s_z} v(p, s_z)\bar{v}(p, s_z) = \frac{\not{p} - m_0c}{2m_0c}.$$

Exercise 3. (15 points) : Gordon identity

Derive the so-called *Gordon identity*

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{P^\mu}{2m_0c} + \frac{i\sigma^{\mu\nu}q_\nu}{2m_0c}\right]u(p),$$

where $P = p' + p$, $q = p' - p$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.