Examples Sheet 4

Symmetries in Physics

Winter 2019/20

Lecturer: PD Dr. G. von Hippel

1. *Characters of SO(3)* (15 P.)

   1. Show that any three-dimensional rotation by an angle φ is conjugate to a rotation by φ around the z-axis.
   2. Considering a basis in which the $(2j+1) \times (2j+1)$ matrix $\rho_j(J_3)$ representing the generator J_3 of rotations around the z-axis is diagonal, show that the character $\chi^{(j)}(\varphi)$ of the $(2j+1)$-dimensional representation of SO(3) can be written as a finite geometric series in $\mathrm{e}^{i\varphi}$.
   3. By a suitable rearrangement of terms reexpress this as
   $$\chi^{(j)}(\varphi) = \frac{\sin\left[\left(j+\frac{1}{2}\right)\varphi\right]}{\sin\left(\frac{1}{2}\varphi\right)} .$$
   4. Using suitable expressions for the characters, show that the Clebsch-Gordan series for SO(3) is
   $$j_1 \otimes j_2 = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} j$$
   with no multiplicities.

2. *Character tables for small groups* (10 P.)

   1. Determine the character tables for all finite groups with no more than six elements. [*Hint:* It's very easy for all but one of them.]
   2. Explicitly check the sums over representations and conjugacy classes (rows and columns) involving the characters and dimensions of the irreps.

3. *Clebsch-Gordan coefficients for SU(2)* (5 P.)

   By repeated application of $J_\pm = j_{1,\pm} + j_{2,\pm}$ calculate the Clebsch-Gordan coefficients for

   1. $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$,
   2. $1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$,
   3. $1 \otimes 1 = 2 \oplus 1 \oplus 0$,

   where the SU(2) irreps are labelled by the value of j corresponding to the eigenvalue $j(j+1)$ of the Casimir operator $\boldsymbol{J}^2$.

4. *Young tableaux for SU(3)* (10 P.)

Using Young tableaux, show the Clebsch-Gordan series

1. $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$,
2. $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$,
3. $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$,

where the SU(3) irreps are labelled by their dimensions.