3) DIRAC PARTICLE IN CENTRAL POTENTIAL

- STATIONARY STATES IN CENTRAL POTENTIAL $V(r)$

e.g. H-ATOM $V(r) = -\frac{e^2 \hbar c}{4\pi r} = -\frac{\alpha}{r}\hbar c$

$\alpha \simeq \frac{1}{137}$

$$\left(c\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 + V(r) \right) \Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

$$\hat{\bar{p}} = -i\hbar \bar{\nabla}$$

$\downarrow$ FOR STATIONARY SOLUTION $\Psi(\bar{r}, t) = e^{-\frac{i}{\hbar}Et} \psi(\bar{r})$

(V DOES NOT DEPEND ON t)

$$\left(c\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 + V(r) \right) \psi(\bar{r}) = E \psi(\bar{r})$$

$$\underline{\underline{H_D \psi(\bar{r}) = E \psi(\bar{r})}}$$

DIRAC HAMILTONIAN

$$\boxed{H_D = c\bar{\alpha}.\hat{\bar{p}} + \beta m_0 c^2 + V(r)}$$

$$\bar{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

- ORBITAL ANGULAR MOMENTUM

$$\hat{\bar{L}} = \bar{r} \times \hat{\bar{p}}$$

↳ FOR SCHRÖDINGER EQ. (SPINLESS PARTICLE IN CENTRAL POTENTIAL)

↓

$\bar{L}^2, L_z$ COMMUTE WITH H_S

↳ SCHRÖDINGER

⇓

ℓ^2, ℓ_z ARE GOOD QUANTUM NUMBERS FOR SCHRÖDINGER THEORY

↳ IN DIRAC THEORY

$$[H_D, L_i]$$

$$= \varepsilon_{ijk} [H_D, r_j \hat{p}_k]$$

$$= \varepsilon_{ijk} [c \alpha_\ell \hat{p}_\ell, r_j \hat{p}_k]$$

$[V(r), L_i] = 0$

$$= c\, \varepsilon_{ijk} \alpha_\ell \underbrace{[\hat{p}_\ell, r_j]}_{-i\hbar \delta_{j\ell}} \hat{p}_k$$

AS $[\hat{p}_\ell, \hat{p}_k] = 0$

$$= -i\hbar c\, \varepsilon_{ijk} \alpha_j \hat{p}_k$$

$$= -i\hbar c \left(\bar{\alpha} \times \hat{\bar{p}}\right)_i \neq 0.$$

∴ $\bar{L}$ DOES NOT COMMUTE WITH H_D

↳ ℓ_z IS NOT GOOD QUANTUM NUMBER

- **SPIN** (INTRINSIC ANGULAR MOMENTUM)

$$\bar{S} = \frac{\hbar}{2}\bar{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$$

$$[H_D, S_i] = \frac{\hbar}{2}\left[c\,\bar{\alpha}.\hat{\bar{p}}, \bar{\Sigma}\right]$$

$$= \frac{\hbar c}{2}\left\{\begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ \bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} - \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}\begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ \bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix}\right\}$$

$$= \frac{\hbar c}{2}\begin{pmatrix} 0 & [\bar{\sigma}.\hat{\bar{p}}, \sigma_i] \\ [\bar{\sigma}.\hat{\bar{p}}, \sigma_i] & 0 \end{pmatrix}$$

$$\downarrow \qquad [\sigma_k, \sigma_\ell] = 2i\,\varepsilon_{k\ell m}\sigma_m$$

$$= \frac{\hbar c}{2}\begin{pmatrix} 0 & 2i\,\varepsilon_{kij}\,\hat{p}_k\sigma_j \\ 2i\,\varepsilon_{kij}\,\hat{p}_k\sigma_j & 0 \end{pmatrix}$$

$$= i\hbar c\,\varepsilon_{ijk}\,\hat{p}_k\begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$= i\hbar c\,\varepsilon_{ijk}\,\hat{p}_k\,\alpha_j$$

$$= i\hbar c\left(\bar{\alpha}\times\hat{\bar{p}}\right)_i \neq 0$$

$\therefore$ $\bar{S}$ DOES <u>NOT</u> COMMUTE WITH H_D

$\hookrightarrow$ S_z IS NOT GOOD QUANTUM NUMBER.

- TOTAL ANGULAR MOMENTUM

$$\bar{J} = \bar{L} + \bar{S}$$

$$[H_D, J_i] = [H_D, L_i] + [H_D, S_i]$$

$$= 0$$

$$\Downarrow$$

ONLY TOTAL ANGULAR MOMENTUM IS CONSERVED BY DIRAC EQ. (FOR PARTICLE IN CENTRAL POTENTIAL)

IN DIRAC EQ. THERE IS A COUPLING BETWEEN SPIN $\bar{S}$ AND ORBITAL ANGULAR MOMENTUM $\bar{L}$
$\Rightarrow$ SPIN-ORBIT INTERACTION

$$[H_D, J^2] = 0$$

$$[H_D, J_i] = 0$$

$$[J^2, J_i] = 0$$

BUT $[J_i, J_j] = i\hbar\varepsilon_{ijk} J_k$

$\therefore$ H_D, J^2, J_z COMMUTE SIMULTANEOUSLY

SOLUTIONS OF H_D HAVE GOOD QUANTUM NUMBERS j, j_z

$$J^2 \psi = \hbar^2 j(j+1)\psi$$

$$J_z \psi = \hbar j_z \psi$$

$\rightsquigarrow$ $j = \ell \pm \frac{1}{2}$

FOR GIVEN VALUE OF j, 2 SOLUTIONS

$\hookrightarrow$ WE NEED ANOTHER CONSERVED QUANTITY (GOOD QUANTUM NUMBER) TO DISTINGUISH BOTH CASES.

- CONSIDER OPERATOR $K \equiv \beta(\bar{\Sigma}.\bar{L} + \hbar)$

WILL SHOW THAT $[H_D, K] = 0$

↳ $\bar{\Sigma}.\bar{L}$ SPIN-ORBIT OPERATOR

CAN BE WRITTEN EQUIVALENTLY AS
$$\bar{\Sigma}.\bar{L} = \bar{\Sigma}.(\bar{J} - \tfrac{\hbar}{2}\bar{\Sigma}) = \bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\underbrace{\Sigma^2}_{\Sigma^2 = 3\,\mathbb{1}} = \bar{\Sigma}.\bar{J} - \tfrac{3\hbar}{2}$$

$$K = \beta(\bar{\Sigma}.\bar{L} + \hbar) = \beta\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right)$$

↳ PROOF $[H_D, K] = 0$

$$
\begin{aligned}
&[H_D, K] \\
&= [H_D, \beta]\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right) + \beta\left[H_D, \bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right] \\
&= c[\bar{\alpha}.\hat{\bar{p}}, \beta]\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right) + \beta[H_D, \bar{\Sigma}.\bar{J}]
\end{aligned}
$$

↓ AS $\alpha_i\beta = -\beta\alpha_i$ ↓ AS $[H_D, J_i] = 0$

$$= -2c\beta(\bar{\alpha}.\hat{\bar{p}})\left(\bar{\Sigma}.\bar{J} - \tfrac{\hbar}{2}\right) + \beta[H_D, \Sigma_i]J_i$$

$[H_D, \Sigma_i] = 2ic(\bar{\alpha} \times \hat{\bar{p}})_i$

$$= -2c\beta\begin{pmatrix} 0 & \bar{\sigma}.\hat{\bar{p}} \\ +\bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} J_i$$

$$+ \hbar c\,\beta(\bar{\alpha}.\hat{\bar{p}}) + 2ic\,\beta(\bar{\alpha} \times \hat{\bar{p}}).\bar{J}$$

$$(\bar{\sigma}\cdot\hat{\bar{p}})(\bar{\sigma}\cdot\bar{J}) = \hat{\bar{p}}\cdot\bar{J} + i\,\bar{\sigma}\cdot(\hat{\bar{p}}\times\bar{J})$$

$$\downarrow$$

$$(\bar{\sigma}\cdot\hat{\bar{p}})\,\sigma_i = \hat{p}_i + i\,\varepsilon_{kij}\,\hat{p}_k\,\sigma_j$$

$$[H_D, K]$$

$$= -2c\,\beta\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\hat{\bar{p}}\cdot\bar{J} \cancel{- 2c\,\beta\, i\,\varepsilon_{ijk}\begin{pmatrix}0 & \sigma_j\hat{p}_k\\ \sigma_j\hat{p}_k & 0\end{pmatrix}J_i}$$

$$+ c\hbar\,\beta\,\hat{\bar{p}}\cdot\begin{pmatrix}0 & \bar{\sigma}\\ \bar{\sigma} & 0\end{pmatrix} \cancel{+ 2c\,\beta\, i\,\varepsilon_{ijk}\begin{pmatrix}0 & \sigma_j\hat{p}_k\\ \sigma_j\hat{p}_k & 0\end{pmatrix}J_i}$$

$$= -2c\,\beta\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\hat{\bar{p}}\cdot\bar{J}$$

$$+ 2c\,\beta\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}\hat{\bar{p}}\cdot\begin{pmatrix}\frac{\hbar}{2}\bar{\sigma} & 0\\ 0 & \frac{\hbar}{2}\bar{\sigma}\end{pmatrix}$$

$$= -2c\,\beta\underbrace{\begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}}_{\gamma_5}\cdot\hat{\bar{p}}\cdot\underbrace{\left(\bar{J} - \frac{\hbar}{2}\bar{\Sigma}\right)}_{\bar{L}}$$

$$\downarrow \qquad \hat{\bar{p}}\cdot\bar{L} = 0$$

$$\overset{!}{=} 0$$

↳ PROOF $[K, J_i] = 0$

$$[K, J_i]$$

$$= \left[\beta\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right), J_i\right]$$

$$= [\beta, J_i]\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right) + \beta\left[\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}, J_i\right]$$

$$= \frac{\hbar}{2}\underbrace{[\beta, \Sigma_i]}_{0}\left(\bar{\Sigma}.\bar{J} - \frac{\hbar}{2}\right) + \beta\left[\bar{\Sigma}.\bar{J}, J_i\right]$$

$$= \beta\,\Sigma_j\underbrace{[J_j, J_i]}_{i\hbar\,\varepsilon_{jik}\,J_k} + \beta\underbrace{[\Sigma_j, J_i]}_{\frac{\hbar}{2}[\Sigma_j, \Sigma_i] = \frac{\hbar}{2}2i\,\varepsilon_{jik}\Sigma_k} J_j$$

$$= i\hbar\,\varepsilon_{jik}\,\beta\,\Sigma_j\,J_k + i\hbar\,\varepsilon_{jik}\,\beta\,\Sigma_k\,J_j$$

$$\overset{!}{=} 0$$

$\therefore$ $\boxed{H_D, J^2, J_z, K \text{ FORM A SET OF SIMULTANEOUSLY COMMUTING OPERATORS}}$

$\Downarrow$

HAVE COMMON EIGENSTATES

- EIGENVALUES OF $K = \beta(\bar{\Sigma}.\bar{L} + \hbar)$

↳ DEFINE EIGENVALUE OF K AS $-\hbar\kappa$

$$\underline{\underline{K\psi \equiv -\hbar\kappa\,\psi}}$$

↳ DETERMINE K^2

$$K^2 = \beta(\bar{\Sigma}.\bar{L} + \hbar)\,\beta(\bar{\Sigma}.\bar{L} + \hbar)$$

$$\Big\downarrow\ \beta\Sigma_i = \Sigma_i\beta$$

$$= (\Sigma_i L_i)(\Sigma_j L_j) + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$\Big\downarrow\ \Sigma_i\Sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\Sigma_k$$

$$= L^2 + i\,\varepsilon_{ijk} L_i L_j \Sigma_k + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$= L^2 + \frac{i}{2}\varepsilon_{ijk}\underbrace{(L_i L_j - L_j L_i)}_{i\hbar\,\varepsilon_{ij\ell}L_\ell}\Sigma_k + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$\Big\downarrow\ \varepsilon_{ijk}\varepsilon_{ij\ell} = 2\delta_{k\ell}$$

$$= L^2 - \hbar(\bar{\Sigma}.\bar{L}) + 2\hbar(\bar{\Sigma}.\bar{L}) + \hbar^2$$

$$= L^2 + 2(\bar{S}.\bar{L}) + \hbar^2$$

$$= (\bar{L} + \bar{S})^2 - \bar{S}^2 + \hbar^2 \qquad \bar{S}^2 = \frac{\hbar^2}{4}\Sigma^2 = \frac{3\hbar^2}{4}\mathbb{1}$$

$$= J^2 + \frac{\hbar^2}{4}\mathbb{1}$$

$$\underline{\underline{K^2 = J^2 + \frac{\hbar^2}{4}\mathbb{1}}}$$

FOR EIGENSTATE OF $\bar{J}$

$$J^2 \psi = \hbar^2 j(j+1)\psi$$

$$\downarrow$$

$$K^2 \psi = \left(\bar{J}^2 + \frac{\hbar^2}{4}\right)\psi$$

$$= \hbar^2\left(j(j+1) + \frac{1}{4}\right)\psi$$

$$= \hbar^2\left(j + \frac{1}{2}\right)^2 \psi$$

$$= \hbar^2 K^2 \psi$$

$$\boxed{|K| = j + \frac{1}{2}.}$$

$$K = \pm\left(j + \frac{1}{2}\right)$$

↳ SIGN OF K

$$K\psi = \mp\hbar\left(j + \frac{1}{2}\right)\psi$$

$$\beta\begin{pmatrix} \bar{\sigma}.\bar{L} + \hbar & 0 \\ 0 & \bar{\sigma}.\bar{L} + \hbar \end{pmatrix}\psi = \mp\hbar\left(j + \frac{1}{2}\right)\psi$$

$$\downarrow \quad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

WITH ψ_A, ψ_B : 2×1 PAULI SPINORS

$$\begin{pmatrix} \bar{\sigma}.\bar{L} + \hbar & 0 \\ 0 & -\bar{\sigma}.\bar{L} - \hbar \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \mp\hbar\left(j + \frac{1}{2}\right)\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\begin{cases} (\bar{\sigma}.\bar{L})\,\psi_A = \mp\hbar\left(j+\frac{1}{2}\pm 1\right)\psi_A \\ -(\bar{\sigma}.\bar{L})\,\psi_B = \mp\hbar\left(j+\frac{1}{2}\mp 1\right)\psi_B \end{cases}$$

FOR $\underline{K = +(j+\frac{1}{2})}$ $\begin{cases} (\bar{\sigma}.\bar{L})\,\psi_A = -\hbar\,(j+\frac{3}{2})\,\psi_A \\ (\bar{\sigma}.\bar{L})\,\psi_B = \hbar\,(j-\frac{1}{2})\,\psi_B \end{cases}$

$\underline{K = -(j+\frac{1}{2})}$ $\begin{cases} (\bar{\sigma}.\bar{L})\,\psi_A = \hbar\,(j-\frac{1}{2})\,\psi_A \\ (\bar{\sigma}.\bar{L})\,\psi_B = -\hbar\,(j+\frac{3}{2})\,\psi_B \end{cases}$

ψ_A, ψ_B ARE EIGENSTATES OF $\bar{\sigma}.\bar{L}$
BUT WITH DIFFERENT EIGENVALUES

$\hookrightarrow$ $L^2 = J^2 - \hbar\,\bar{\Sigma}.\bar{L} - \frac{3}{4}\hbar^2$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2\left(j(j+1) - \frac{3}{4}\right)\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} - \hbar \begin{pmatrix} \bar{\sigma}.\bar{L}\ \psi_A \\ \bar{\sigma}.\bar{L}\ \psi_B \end{pmatrix}$$

FOR $\underline{K = +(j+\frac{1}{2})}$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2\left[j(j+1) - \frac{3}{4}\right]\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} + \hbar^2 \begin{pmatrix} (j+\frac{3}{2})\,\psi_A \\ -(j-\frac{1}{2})\,\psi_B \end{pmatrix}$$

$$= \hbar^2 \begin{pmatrix} (j^2 + 2j + \frac{3}{4})\,\psi_A \\ (j^2 - \frac{1}{4})\,\psi_B \end{pmatrix}$$

DENOTE $\underline{\underline{\ell_\pm \equiv j \pm \frac{1}{2}}}$

$$\ell_+(\ell_+ + 1) = j^2 + 2j + \frac{3}{4}$$
$$\ell_-(\ell_- + 1) = j^2 - \frac{1}{4}$$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} \ell_+(\ell_+ + 1)\,\psi_A \\ \ell_-(\ell_- + 1)\,\psi_B \end{pmatrix}$$

ψ_A, ψ_B ARE EIGENSTATES OF L^2
BUT WITH DIFFERENT EIGENVALUES.
$\hookrightarrow$ DIRAC SPINOR NOT EIGENSTATE OF L^2

FOR $K = -(j + \frac{1}{2})$

$$L^2 \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \hbar^2 \begin{pmatrix} \ell_-(\ell_- + 1)\,\psi_A \\ \ell_+(\ell_+ + 1)\,\psi_B \end{pmatrix}$$

$\therefore$
$$K = +(j + \tfrac{1}{2}) > 0 \Rightarrow \ell_A = \ell_+ , \ \ell_B = \ell_-$$
$$K = -(j + \tfrac{1}{2}) < 0 \Rightarrow \ell_A = \ell_- , \ \ell_B = \ell_+$$

NOTE SIGN OF K INDICATES WHETHER $\bar{L}$ & $\bar{\sigma}$ ARE // OR ANTI-//
(OPPOSITE FOR UPPER COMPONENT ψ_A
& LOWER COMPONENT ψ_B)

$$\begin{cases} (\bar{\sigma}\cdot\bar{L})\,\psi_A = -\hbar(K+1)\,\psi_A \\ (\bar{\sigma}\cdot\bar{L})\,\psi_B = +\hbar(K-1)\,\psi_B \end{cases}$$

• SOLUTION OF DIRAC EQ. IN CENTRAL POTENTIAL.

$\hookrightarrow$ $$H_D \psi = E\psi \qquad \psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\begin{pmatrix} 0 & c\,\bar{\sigma}.\hat{\bar{p}} \\ c\,\bar{\sigma}.\hat{\bar{p}} & 0 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} (E - V(r) - m_0c^2)\,\psi_A \\ (E - V(r) + m_0c^2)\,\psi_B \end{pmatrix}$$

$$\begin{cases} c\,(\bar{\sigma}.\hat{\bar{p}})\,\psi_B = (E - V(r) - m_0c^2)\,\psi_A \\ c\,(\bar{\sigma}.\hat{\bar{p}})\,\psi_A = (E - V(r) + m_0c^2)\,\psi_B \end{cases}$$

DENOTE $$\boxed{\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} g(r)\,\varphi_{jl_Am} \\ i f(r)\,\varphi_{jl_Bm} \end{pmatrix}}$$

ψ HAS GOOD j, m, K

$$\begin{cases} J^2\psi = \hbar^2 j(j+1)\psi \\ J_z\psi = \hbar m\,\psi \\ K\psi = -\hbar\kappa\,\psi \end{cases} \quad \text{WITH } \kappa = \pm\left(j+\tfrac{1}{2}\right)$$

FOR $\kappa > 0 \;\rightarrow\; l_A = j + \frac{1}{2}$, $l_B = j - \frac{1}{2}$

$\kappa < 0 \;\rightarrow\; l_A = j - \frac{1}{2}$, $l_B = j + \frac{1}{2}$

φ_{jlm} DEPENDS ONLY ON ANGLES

$\hookrightarrow$ EIGENSTATES OF J^2, J_z, L^2

PAULI SPINOR $\downarrow$

$$\boxed{\varphi_{jlm}(\bar{e}_r) = \sum_{m_l}\sum_{m_s} \langle l\, m_l, \tfrac{1}{2} m_s \,|\, j\, m\rangle\, Y_{lm_l}(\bar{e}_r)\,\chi_{m_s}}$$

$$\hookrightarrow \quad \bar{\sigma}.\hat{\bar{p}} = \underbrace{(\bar{\sigma}.\bar{e}_r)(\bar{\sigma}.\bar{e}_r)}_{1}(\bar{\sigma}.\hat{\bar{p}}) \qquad \bar{e}_r = \frac{\bar{r}}{r}$$

$$= (\bar{\sigma}.\bar{e}_r)\left\{ \bar{e}_r.\hat{\bar{p}} + i\,\bar{\sigma}.(\bar{e}_r \times \hat{\bar{p}})\right\}$$

$$\downarrow \qquad \bar{e}_r.\hat{\bar{p}} = -i\hbar\, \bar{e}_r.\bar{\nabla}$$

$$= -i\hbar \frac{\partial}{\partial r}$$

$$= (\bar{\sigma}.\bar{e}_r)\left\{ -i\hbar\frac{\partial}{\partial r} + \frac{i}{r}\,\bar{\sigma}.\underbrace{(\bar{r}\times\hat{\bar{p}})}_{\bar{L}}\right\}$$

$$\underline{\underline{\bar{\sigma}.\hat{\bar{p}} = (\bar{\sigma}.\bar{e}_r)\left\{-i\hbar\frac{\partial}{\partial r} + \frac{i}{r}\,\bar{\sigma}.\bar{L}\right\}}}$$

NOTE $\bar{\sigma}.\bar{e}_r$ & $\bar{\sigma}.\bar{L}$ ONLY ACT ON ANGULAR PARTS OF WAVE FUNCTION.

$$\begin{cases} c(\bar{\sigma}.\hat{\bar{p}})\,\psi_B = (E - V(r) - m_0c^2)\,\psi_A \\ c(\bar{\sigma}.\hat{\bar{p}})\,\psi_A = (E - V(r) - m_0c^2)\,\psi_B \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} c(\bar{\sigma}.\bar{e}_r)\left\{\hbar f'(r) - \hbar\frac{f(r)}{r}(K-1)\right\}\varphi_{j\ell_B m} \\ = (E - V(r) - m_0c^2)\,g(r)\,\varphi_{j\ell_A m} \\ c(\bar{\sigma}.\bar{e}_r)\left\{-i\hbar\, g'(r) - i\hbar\frac{g(r)}{r}(K+1)\right\}\varphi_{j\ell_A m} \\ = (E - V(r) + m_0c^2)\,i f(r)\,\varphi_{j\ell_B m} \end{cases}$$

$\hookrightarrow \quad (\bar{\sigma}.\bar{e}_r)\, \varphi_{j\ell m}$

$\rightsquigarrow \quad [\bar{\sigma}.\bar{e}_r\, ,\, J_i]$

$$= \frac{1}{r}\left[\bar{\sigma}.\bar{r}\, ,\, (\bar{r} \times \hat{\bar{p}})_i + \frac{\hbar}{2}\sigma_i\right]$$

$$= \frac{1}{r}\Big\{\varepsilon_{ijk}\, \sigma_\ell\, \underbrace{[r_\ell\, ,\, r_j \hat{\bar{p}}_k]}_{r_j \underbrace{[r_\ell\, ,\, \hat{\bar{p}}_k]}_{i\hbar\delta_{k\ell}}} + \frac{\hbar}{2} r_\ell\, \underbrace{[\sigma_\ell\, ,\, \sigma_i]}_{2i\,\varepsilon_{\ell ij}\sigma_j}\Big\}$$

$$= \frac{i\hbar}{r}\left\{\varepsilon_{ijk}\, r_j \sigma_k + \varepsilon_{ij\ell}\, \sigma_j r_\ell\right\}$$

$$= 0$$

$\therefore \quad (\bar{\sigma}.\bar{e}_r)\, \varphi_{j\ell m}$ IS EIGENSTATE OF J^2, J_z WITH QUANTUM NUMBERS j, m

$\rightsquigarrow$ UNDER SPATIAL INVERSION

$\bar{r} \to -\bar{r}$

$\bar{\sigma} \to \bar{\sigma}$

$(\bar{\sigma}.\bar{e}_r) \longrightarrow -(\bar{\sigma}.\bar{e}_r)$ TRANSFORMS AS PSEUDOSCALAR

$\varphi_{j\ell m}$: PARITY $(-1)^\ell$

$(\bar{\sigma}.\bar{e}_r)$ CHANGES PARITY OF STATE

$$(\bar{\sigma}.\bar{e}_r)\, \underbrace{\varphi_{j\ell_A m}}_{(-1)^{\ell_A}} = C\, \underbrace{\varphi_{j\ell_B m}}_{(-1)^{\ell_B} = -(-1)^{\ell_A}}$$

PARITY

AS $\underset{\sim}{\ell_B = \ell_A \mp 1}$

$$(\bar{\sigma}\cdot\bar{e}_r)^2 = 1 \Rightarrow C^2 = 1 \Rightarrow C = \pm 1$$

$$(\bar{\sigma}\cdot\bar{e}_r)\,\varphi_{j l_A m} = \underbrace{-}_{\text{PHASE CONVENTION}}\,\varphi_{j l_B m}$$

CHECK FOR $l_A = 0$

$$(\bar{\sigma}\cdot\bar{e}_r) = \sin\theta\cos\phi\,\sigma_x + \sin\theta\sin\phi\,\sigma_y + \cos\theta\,\sigma_z$$

$$\downarrow \qquad S_\pm = \frac{\hbar}{2}(\sigma_x \pm i\sigma_y)$$

$$\hbar\sigma_x = S_+ + S_-$$

$$\hbar\sigma_y = -i(S_+ - S_-)$$

$$\hbar(\bar{\sigma}\cdot\bar{e}_r) = \sin\theta\, e^{-i\phi} S_+ + \sin\theta\, e^{+i\phi} S_- + 2\cos\theta\, S_z$$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$\downarrow \qquad Y_{1,-1} = +\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\hbar(\bar{\sigma}\cdot\bar{e}_r) = 2\sqrt{\frac{2\pi}{3}}\,Y_{1,-1}\,S_+ - 2\sqrt{\frac{2\pi}{3}}\,Y_{1,+1}\,S_- + 2\sqrt{\frac{4\pi}{3}}$$

SPECIAL CASE $l_A = 0$, $l_B = 1$, $j = +\frac{1}{2}$, $m = +\frac{1}{2}$

$$\varphi_{\frac{1}{2}\,0\,\frac{1}{2}}(\bar{e}_r) = Y_{00}(\bar{e}_r)\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \chi_{+\frac{1}{2}}$$

$$\varphi_{\frac{1}{2}\,1\,\frac{1}{2}}(\bar{e}_r) = \sum_{m_l m_s} \langle 1\, m_l, \tfrac{1}{2}\, m_s | \tfrac{1}{2} + \tfrac{1}{2}\rangle\, Y_{1 m_l}(\bar{e}_r)\,\chi_{m_s}$$

$$\varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r) = \langle 1\,0, \tfrac{1}{2} + \tfrac{1}{2} \mid \tfrac{1}{2} + \tfrac{1}{2} \rangle Y_{10}(\bar{e}_r)\, \chi_{+\frac{1}{2}}$$

$$+ \langle 1 + 1, \tfrac{1}{2} - \tfrac{1}{2} \mid \tfrac{1}{2} + \tfrac{1}{2} \rangle Y_{1+1}(\bar{e}_r)\, \chi_{-\frac{1}{2}}$$

$$= -\sqrt{\tfrac{1}{3}}\, Y_{10}(\bar{e}_r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\tfrac{2}{3}}\, Y_{11}(\bar{e}_r) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\therefore\ \hbar\, (\bar{\sigma}\cdot\bar{e}_r)\, \varphi_{\frac{1}{2}0\frac{1}{2}}(\bar{e}_r)$$

$$= -2\sqrt{\frac{2\pi}{3}}\, \frac{1}{\sqrt{4\pi}}\, Y_{1,+1}(\bar{e}_r)\, \underbrace{S_-\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$+ 2\sqrt{\frac{4\pi}{3}}\, \frac{1}{\sqrt{4\pi}}\, Y_{1,0}(\bar{e}_r)\, \underbrace{S_z\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$= \hbar \left\{ -\sqrt{\tfrac{2}{3}}\, Y_{11}(\bar{e}_r) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{\tfrac{1}{3}}\, Y_{10}(\bar{e}_r) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$= -\hbar\, \varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r)$$

$$\therefore\ \underline{\underline{(\bar{\sigma}\cdot\bar{e}_r)\, \varphi_{\frac{1}{2}0\frac{1}{2}}(\bar{e}_r) = -\varphi_{\frac{1}{2}1\frac{1}{2}}(\bar{e}_r)}}$$

↳ COUPLED RADIAL EQUATIONS FOR f, g

$$\begin{cases} -\hbar c \left\{ f' - \frac{f}{r}(K-1) \right\} = \left(E - V(r) - m_0c^2\right) g \\ -\hbar c \left\{ -g' - \frac{g}{r}(K+1) \right\} = \left(E - V(r) + m_0c^2\right) f \end{cases}$$

$$\downarrow \qquad g(r) \equiv \frac{G(r)}{r} \,, \quad f(r) \equiv \frac{F(r)}{r}$$

$$\boxed{\begin{cases} \frac{\partial F}{\partial r} - K\frac{F}{r} = \frac{1}{\hbar c}\left(-E + V(r) + m_0c^2\right) G \\ \frac{\partial G}{\partial r} + K\frac{G}{r} = \frac{1}{\hbar c}\left(E - V(r) + m_0c^2\right) F \end{cases}}$$

↳ TOTAL SOLUTION

$$\psi = \begin{pmatrix} \frac{G(r)}{r} \varphi_{j l_A m}(\bar{e}_r) \\ i \frac{F(r)}{r} \varphi_{j l_B m}(\bar{e}_r) \end{pmatrix}$$

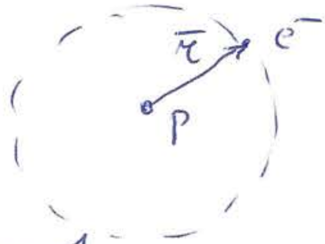
FOR $K = +(j + \frac{1}{2}) \qquad l_A = j + \frac{1}{2} \,, \; l_B = j - \frac{1}{2}$

$K = -(j + \frac{1}{2}) \qquad l_A = j - \frac{1}{2} \,, \; l_B = j + \frac{1}{2}$

HYDROGEN ATOM IN DIRAC THEORY

↳ BOUND STATES IN

COULOMB POTENTIAL OF e^- ORBITING AROUND PROTON



$$\boxed{\begin{aligned} V(r) &= -\frac{Ze^2 \hbar c}{4\pi r} \\ &= -\frac{Z\alpha}{r} \hbar c \end{aligned}}$$

$$\alpha \equiv \frac{e^2}{4\pi} \simeq \frac{1}{137}$$

↳ FINE-STRUCTURE CONSTANT

FOR PROTON $Z = +1$
He^+ $Z = +2, \dots$

↳ INTRODUCE

$$k_1 \equiv \frac{1}{\hbar c}\left(E + m_0 c^2\right)$$

$$k_2 \equiv \frac{1}{\hbar c}\left(-E + m_0 c^2\right)$$

FOR BOUND STATES $E < m_0 c^2 \implies k_2 > 0$

$$\begin{cases} \dfrac{\partial F}{\partial r} - K \dfrac{F}{r} = \left(k_2 - \dfrac{Z\alpha}{r}\right) G \\[2ex] \dfrac{\partial G}{\partial r} + K \dfrac{G}{r} = \left(k_1 + \dfrac{Z\alpha}{r}\right) F \end{cases}$$

DIMENSIONLESS VARIABLE

$$\rho \equiv 2\sqrt{k_1 k_2}\; r = \frac{2}{\hbar c}\left(m_0^2 c^4 - E^2\right)^{1/2} r$$

$$\begin{cases} \dfrac{\partial F}{\partial \rho} - K \dfrac{F}{\rho} = \left(\dfrac{1}{2}\sqrt{\dfrac{k_2}{k_1}} - \dfrac{Z\alpha}{\rho}\right) G \\[2ex] \dfrac{\partial G}{\partial \rho} + K \dfrac{G}{\rho} = \left(\dfrac{1}{2}\sqrt{\dfrac{k_1}{k_2}} + \dfrac{Z\alpha}{\rho}\right) F \end{cases}$$

↳ ASYMPTOTIC FORM $\rho \to \infty$

$$\frac{\partial F}{\partial \rho} \simeq \frac{1}{2}\sqrt{\frac{k_2}{k_1}}\, G \qquad \Rightarrow \qquad \frac{\partial^2 F}{\partial \rho^2} = \frac{1}{4} F$$

$$\frac{\partial G}{\partial \rho} \simeq \frac{1}{2}\sqrt{\frac{k_1}{k_2}}\, F \qquad\qquad \frac{\partial^2 G}{\partial \rho^2} = \frac{1}{4} G$$

$$\therefore \quad F(\rho) \underset{\rho \to \infty}{\sim} e^{-\rho/2}$$

$$G(\rho) \underset{\rho \to \infty}{\sim} e^{-\rho/2}$$

↳ SOLUTION IN POWER SERIES FORM

$$F(\rho) = \sqrt{k_2}\, e^{-\rho/2} \sum_{m=0}^{\infty} a_m \rho^{m+\gamma}$$

$$G(\rho) = \sqrt{k_1}\, e^{-\rho/2} \sum_{m=0}^{\infty} b_m \rho^{m+\gamma}$$

BEHAVIOR FOR $\rho \to 0$ $\qquad F(\rho) \sim \rho^{\gamma}$

$G(\rho) \sim \rho^{\gamma}$

↳ RECURSION RELATIONS

$$\frac{\partial F}{\partial \rho} = -\frac{1}{2} F + \sqrt{k_2}\, e^{-\rho/2} \sum_{m=0}^{\infty} a_m (m+\gamma) \rho^{m+\gamma-1}$$

$$\frac{\partial G}{\partial \rho} = -\frac{1}{2} G + \sqrt{k_1}\, e^{-\rho/2} \sum_{m=0}^{\infty} b_m (m+\gamma) \rho^{m+\gamma-1}$$

$$\hookrightarrow \frac{\partial F}{\partial \rho} - K\frac{F}{\rho} = \sqrt{k_2}\, e^{-\rho/2}\sum_{m=0}^{\infty}\left\{-\tfrac{1}{2}a_m\,\rho^{m+\gamma} - K\,a_m\,\rho^{m+\gamma-1} + a_m(m+\gamma)\,\rho^{m+\gamma-1}\right\}$$

$\downarrow$ IN 1° TERM : INTRODUCE $\underline{a_{-1} = 0}$

& EXTEND SUM $-1 \rightarrow +\infty$

$$= \sqrt{k_2}\, e^{-\rho/2}\left\{-\sum_{m=-1}^{\infty}\tfrac{1}{2}a_m\,\rho^{m+\gamma} + \sum_{m=0}^{\infty}a_m\,\rho^{m+\gamma-1}\left(-K+m+\gamma\right)\right\}$$

$\downarrow$

$m' = m-1$

$$= \sqrt{k_2}\, e^{-\rho/2}\sum_{m=-1}^{\infty}\rho^{m+\gamma}\left\{-\tfrac{1}{2}a_m + a_{m+1}\left(m+1+\gamma-K\right)\right\}$$

$$\hookrightarrow \left(\frac{1}{2}\sqrt{\frac{k_2}{k_1}} - \frac{Z\alpha}{\rho}\right)G = \sqrt{k_2}\, e^{-\rho/2}\sum_{m=-1}^{\infty}\left\{\tfrac{1}{2}b_m - Z\alpha\frac{k_1}{\sqrt{k_1k_2}}\,b_{m+1}\right\}\rho^{m+\gamma}$$

WITH $\underline{b_{-1} = 0}$

$\therefore$ EQUATE BOTH SIDES

$$\boxed{-\tfrac{1}{2}a_m + \left(m+1+\gamma-K\right)a_{m+1} = \tfrac{1}{2}b_m - \frac{Z\alpha\left(E+m_0c^2\right)}{\hbar c\sqrt{k_1k_2}}\,b_{m+1}}$$

ANALOGOUSLY $a \leftrightarrow b$, $K \leftrightarrow -K$, $Z\alpha \rightarrow -Z\alpha$, $k_1 \leftrightarrow k_2$

$$\boxed{-\tfrac{1}{2}b_m + \left(m+1+\gamma+K\right)b_{m+1} = \tfrac{1}{2}a_m + \frac{Z\alpha\left(-E+m_0c^2\right)}{\hbar c\sqrt{k_1k_2}}\,a_{m+1}}$$

↳ TERM FOR $m = -1$

$$a_{-1} = b_{-1} = 0$$

$$\begin{cases} (\gamma - K)\, a_0 = - \dfrac{Z\alpha\,(E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}}\, b_0 \\ (\gamma + K)\, b_0 = + \dfrac{Z\alpha\,(-E + m_0 c^2)}{\hbar c \sqrt{k_1 k_2}}\, a_0 \end{cases}$$

$$\Downarrow$$

$$(\gamma^2 - K^2) = -(Z\alpha)^2 \frac{m_0^2 c^4 - E^2}{(\hbar c)^2 k_1 k_2}$$

$$= -(Z\alpha)^2$$

CONDITION TO HAVE A REGULAR SOLUTION AT $r = 0$

$$\gamma = \sqrt{K^2 - (Z\alpha)^2}$$

$$\boxed{\gamma = \sqrt{(j + \tfrac{1}{2})^2 - (Z\alpha)^2}}$$

↳ ADD EQUATIONS & $m \to m-1$

$$\left(m + \gamma + \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_m + b_m)$$
$$= (a_{m-1} + b_{m-1}) + \left(K + \frac{Z\alpha m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)(a_m - b_m)$$

↳ SUBTRACT EQUATIONS & $m \to m-1$

$$\left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_m - b_m)$$
$$= \left(K - \frac{Z\alpha m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)(a_m + b_m)$$

$$\therefore \quad (a_m - b_m) = \frac{\left(K - \frac{Z\alpha m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)}{\left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)}(a_m + b_m)$$

↳ PLUG INTO 1° EQ.

$$(a_m + b_m)\left[(m+\gamma)^2 - \frac{(Z\alpha)^2 E^2}{(\hbar c)^2 k_1 k_2} - K^2 + \frac{(Z\alpha)^2 m_0^2 c^4}{(\hbar c)^2 k_1 k_2}\right]$$
$$= \left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_{m-1} + b_{m-1})$$

$$\Updownarrow$$

$$(a_m + b_m)\left[m^2 + 2m\gamma + \gamma^2 + (Z\alpha)^2 - K^2\right]$$
$$= \left(m + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)(a_{m-1} + b_{m-1})$$

$$\Downarrow \qquad \gamma^2 = K^2 - (Z\alpha)^2$$

$$\left(a_m + b_m\right) = \frac{m - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)}{m\,(2\gamma + m)} \left(a_{m-1} + b_{m-1}\right)$$

$$= \frac{\left[1 - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)\right] \cdots \left[m - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)\right]}{m!\,(2\gamma + 1) \cdots (2\gamma + m)} \left(a_0 + b_0\right)$$

$$\left(a_m - b_m\right) = \frac{\left(K - \frac{Z\alpha\, m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)}{\left[m - \left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right)\right]} \left(a_m + b_m\right)$$

$$\downarrow \quad \left(a_0 + b_0\right) = \frac{\left(\gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)}{\left(K - \frac{Z\alpha\, m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)} \left(a_0 - b_0\right)$$

$$\left(a_m - b_m\right) = (-1)^m \frac{\left[\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right]\left[\left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right) - 1\right] \cdots \left[\left(\frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma\right) - m + 1\right]}{m!\,(2\gamma + 1) \cdots (2\gamma + m)}$$

$$\cdot \left(a_0 - b_0\right).$$

SERIES CAN BE EXPRESSED THROUGH CONFLUENT HYPERGEOMETRIC FUNCTION

$$ {}_1F_1(a, b; z) \equiv \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

$$= 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots$$

POCHHAMMER SYMBOL $(a)_k = \dfrac{\Gamma(a+k)}{\Gamma(a)}$

↳ FOR NORMALIZABLE SOLUTIONS

SERIES MUST TERMINATE FOR $n = n' \geq 0$

i.e. $a_{n'+1} = b_{n'+1} = 0$

$$\Downarrow$$

$$\underline{\underline{a_{n'} = -b_{n'}}}$$

$$(a_{n'} - b_{n'})\left(n' + \gamma - \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}}\right)$$

$$= (a_{n'} + b_{n'})\left(K - \frac{Z\alpha\, m_0 c^2}{\hbar c\sqrt{k_1 k_2}}\right)$$

$$= 0$$

$$\Downarrow$$

$$\boxed{n' = \frac{Z\alpha E}{\hbar c\sqrt{k_1 k_2}} - \gamma} \geq 0$$

INTEGER
$n' = 0, 1, \ldots$

$$\left(\frac{m_0^2 c^4}{E^2} - 1\right)^{1/2} = \frac{Z\alpha}{(n' + \gamma)}$$

$$E = \frac{m_0 c^2}{\left[1 + \frac{(Z\alpha)^2}{(n'+\gamma)^2}\right]^{1/2}}$$

$$\Downarrow \qquad \gamma = \left[\left(j + \tfrac{1}{2}\right)^2 - (Z\alpha)^2\right]^{1/2}$$

$$\boxed{E = \frac{m_0 c^2}{\left[1 + \frac{(Z\alpha)^2}{\left(n' + \sqrt{\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2}\right)^2}\right]^{1/2}}}$$

$n' = 0, 1, \ldots$

↳ PERTURBATIVE EXPANSION IN $(Z\alpha)$

$$(Z\alpha) \ll 1$$

FOR $Z=1$, $\alpha = \frac{1}{137}$: $Z\alpha \simeq 10^{-2} \ll 1$

$$E \simeq m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{\left(n' + j + \frac{1}{2} - \frac{Z^2\alpha^2}{2(j+\frac{1}{2})}\right)^2} \right]^{-1/2}$$

$$\simeq m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{(n'+j+\frac{1}{2})^2 \left(1 - \frac{(Z\alpha)^2}{(j+\frac{1}{2})(n'+j+\frac{1}{2})}\right)} \right]^{-1/2}$$

$$\simeq m_0 c^2 \left[1 + \frac{(Z\alpha)^2}{(n'+j+\frac{1}{2})^2} + \frac{(Z\alpha)^4}{(j+\frac{1}{2})(n'+j+\frac{1}{2})^3} \right]^{-1/2}$$

$$\downarrow \qquad (1 + ax + bx^2)^{-1/2} \underset{x \ll 1}{\simeq} 1 - \frac{a}{2}x + \frac{1}{2}x^2\left(\frac{3}{4}a^2 - b\right)$$

$$E \simeq m_0 c^2 \left[1 - \frac{1}{2}\frac{(Z\alpha)^2}{(n'+j+\frac{1}{2})^2} + \frac{1}{2}\frac{(Z\alpha)^4}{(n'+j+\frac{1}{2})^4}\left(\frac{3}{4} - \frac{n'+j+\frac{1}{2}}{j+\frac{1}{2}}\right) \right]$$

INTRODUCE PRINCIPAL QUANTUM NUMBER

$$n \equiv n' + j + \frac{1}{2}$$

$n = 1, 2, \ldots$

$n' = 0,\ j = \frac{1}{2} \Rightarrow n = 1$

BINDING ENERGY E_{nj}

$$\boxed{E_{nj} \equiv E - m_0 c^2 \simeq -\frac{1}{2}(m_0 c^2)\frac{(Z\alpha)^2}{n^2} \cdot \Big[\underset{\substack{\uparrow \\ \text{BOHR} \\ \text{THEORY}}}{1} + \underbrace{\frac{(Z\alpha)^2}{n^2}\left(-\frac{3}{4} + \frac{n}{j+\frac{1}{2}}\right)}_{\text{FINE STRUCTURE}} \Big]}$$

IN DIRAC THEORY BINDING ENERGY DEPENDS ON n AND j

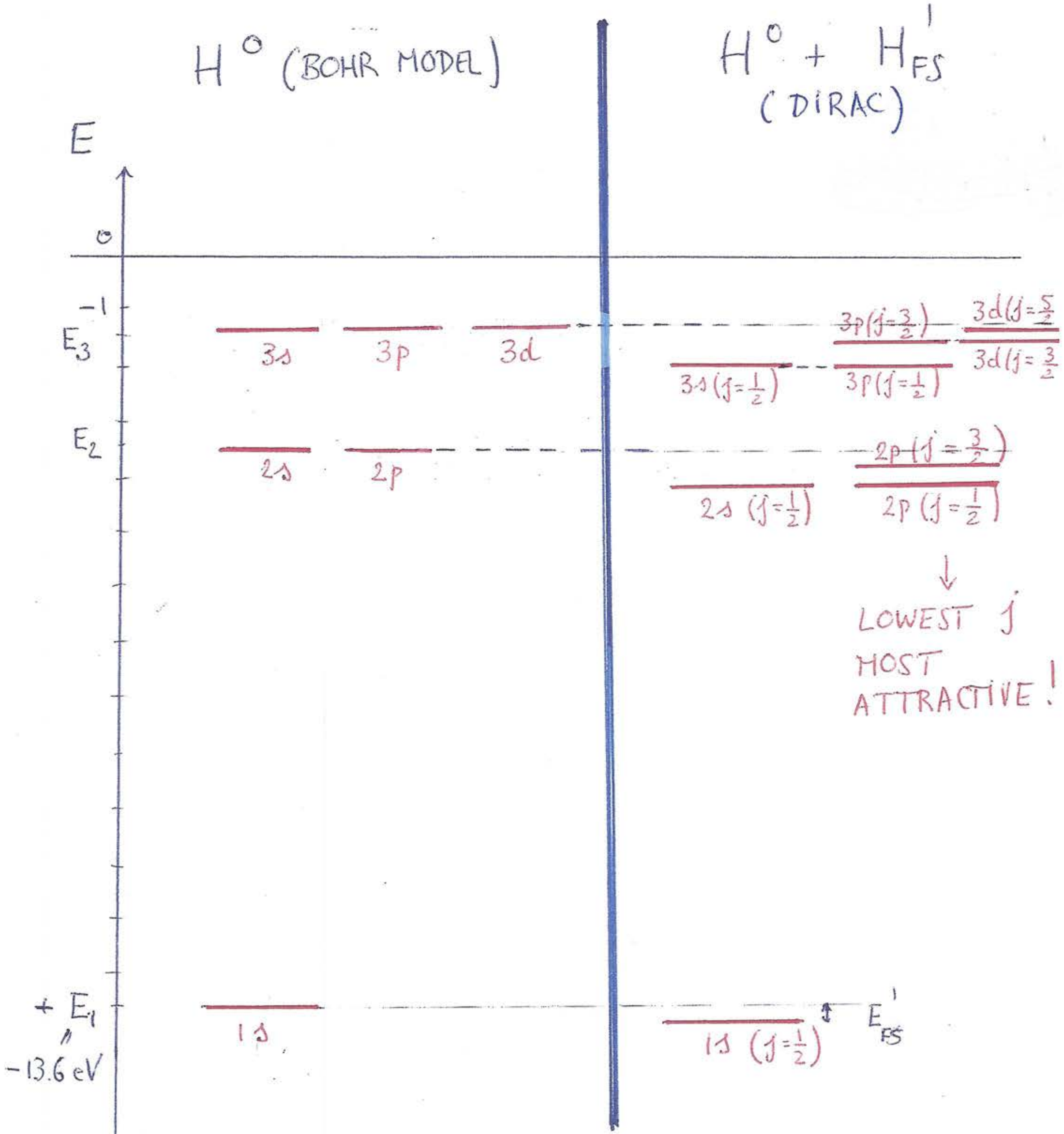
FOR H $(Z = 1)$

$$E_{nj} = -\frac{13.6\ \text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(-\frac{3}{4} + \frac{n}{j+\frac{1}{2}}\right)\right]$$

$$= E_n + E'_{FS} \qquad \text{WITH } E_n = -\frac{13.6\ \text{eV}}{n^2}$$

- $n = 1$, $j = 1/2$: $\quad (E'_{FS})_{1\frac{1}{2}} = -13.6\ \text{eV}\left[1 + \frac{\alpha^2}{4}\right] < -13.6\ \text{eV}$
- $n = 2$, $j = 1/2$: $\quad (E'_{FS})_{2\frac{1}{2}} = -\frac{13.6\ \text{eV}}{4}\left[1 + \frac{\alpha^2}{4}\cdot\frac{5}{4}\right]$
- $n = 2$, $j = 3/2$: $\quad (E'_{FS})_{2\frac{3}{2}} = -\frac{13.6\ \text{eV}}{4}\left[1 + \frac{\alpha^2}{4}\cdot\frac{1}{4}\right]$

FINE - STRUCTURE OF H



TOTAL E

$$\boxed{E = E_n + E'_{FS} = -\frac{13.6\,\text{eV}}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4}\right)\right]}$$

DEPENDS ON n & j

4) QUANTIZATION OF DIRAC FIELD

• DIRAC LAGRANGIAN

↳ DIRAC EQ. $$(i\hbar \gamma^\mu \partial_\mu - m_0 c)\psi = 0$$

WITH $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$

NOTE $\begin{cases} \gamma^{0\dagger} = \gamma^0 \\ \gamma^{i\dagger} = -\gamma^i \end{cases}$ $\quad \gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$ $\quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

↳ ADJOINT FIELD

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0$$

TAKE † OF DIRAC EQ. :

$$-i\hbar(\partial_\mu \psi^\dagger)\gamma^{\mu\dagger} - \psi^\dagger m_0 c = 0$$

$$\Updownarrow$$

$$i\hbar(\partial_\mu \psi^\dagger)\gamma^0\gamma^\mu\gamma^0 + \psi^\dagger m_0 c = 0$$

$\Downarrow$ MULTIPLY BY γ^0 ON RIGHT

$$i\hbar(\partial_\mu \bar{\psi})\gamma^\mu + \bar{\psi} m_0 c = 0$$

LAGRANGIAN

TREAT ψ & $\bar{\psi}$ AS INDEPENDENT FIELDS (COMPLEX VALUED)

DIRAC EQS. FOR ψ & $\bar{\psi}$ CAN BE DERIVED FROM LAGRANGIAN

$$\boxed{\mathcal{L} = c\,\bar{\psi}\left[i\hbar\gamma^\mu\partial_\mu - m_0 c\right]\psi}$$

⇝ EULER-LAGRANGE EQ. FOR $\bar{\psi}$:

$$\frac{\partial \mathcal{L}}{\partial \psi} = c\,\bar{\psi}\,(-m_0 c)$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = c\,\bar{\psi}\,(i\hbar\gamma^\mu)$$

$$\Downarrow$$

EL. EQ. $\quad \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0$

$$\Downarrow$$

$$c\,\bar{\psi}(-m_0 c) - c\left(\partial_\mu\bar{\psi}\right)i\hbar\gamma^\mu = 0$$

$$\therefore \quad i\hbar\left(\partial_\mu\bar{\psi}\right)\gamma^\mu + \bar{\psi}m_0 c = 0 \quad ✓$$

⇝ EULER-LAGRANGE EQ. FOR ψ:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = c\left(i\hbar\gamma^\mu\partial_\mu - m_0 c\right)\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \qquad \therefore \quad i\hbar\gamma^\mu\partial_\mu\psi - m_0 c\,\psi = 0 \quad ✓$$

CONJUGATE MOMENTA:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\hbar \bar{\psi} \gamma^0 = i\hbar \psi^+$$

$$\bar{\pi} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0$$

HAMILTONIAN

$$H = \int d^3\bar{x} \left(\pi \dot{\psi} + \dot{\bar{\psi}} \underbrace{\cancel{\bar{\pi}}}_{0} - \mathcal{L} \right)$$

$$= \int d^3\bar{x} \left(i\hbar \psi^+ \frac{\partial \psi}{\partial t} - c i\hbar \bar{\psi} \gamma^\mu \partial_\mu \psi + m_0 c^2 \bar{\psi}\psi \right)$$

$$= c \int d^3\bar{x}\, \bar{\psi} \left(i\hbar \gamma^0 \partial_0 - i\hbar \gamma^\mu \partial_\mu + m_0 c \right) \psi$$

$$= c \int d^3\bar{x}\, \bar{\psi} \left(-i\hbar \gamma^i \frac{\partial}{\partial x^i} + m_0 c \right) \psi$$

MOMENTUM

4 MOMENTUM $\quad c P^\nu \equiv \int d^3\bar{x} \left\{ c\, \pi\, \partial^\nu \psi - \mathcal{L} g^{0\nu} \right\}$

$\underline{\nu = i}$

$$c P^i = c \int d^3\bar{x}\; \pi (\partial^i \psi)$$

$$P^i = \int d^3\bar{x}\; \psi^+ (-i\hbar \nabla^i) \psi$$

NOTE $\quad \partial^i = \frac{\partial}{\partial x_i} = -\nabla^i = -\frac{\partial}{\partial x^i}$

↳ CHARGE

⇝ CONSIDER GLOBAL PHASE TRANSFORMATION

$$\psi \to e^{i\alpha}\psi \qquad \alpha \text{ CONSTANT REAL NUMBER}$$
$$\bar{\psi} \to \bar{\psi} e^{-i\alpha}$$

⇝ FOR α INFINITESIMAL

$$\psi \to \psi + \underbrace{i(\delta\alpha)\psi}_{\delta\psi}$$
$$\bar{\psi} \to \bar{\psi} \underbrace{- i(\delta\alpha)\bar{\psi}}_{\delta\bar{\psi}}$$

⇝ $\mathcal{L} = c\,\bar{\psi}\left[i\gamma^\mu\partial_\mu - m_0 c\right]\psi$

$\mathcal{L}$ IS INVARIANT UNDER GLOBAL PHASE TF.

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) + \delta\bar{\psi}\underbrace{\cancel{\frac{\partial\mathcal{L}}{\partial\bar{\psi}}}}_{0 \text{ (DIRAC EQ.)}}$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi\right)$$

$$= 0$$

$\Updownarrow$

CONSERVED CURRENT :

$$\partial_\mu \left(\bar{\psi} (i\hbar \gamma^\mu) \cdot (i(\delta\alpha) \psi) \right) = 0$$

$$\partial_\mu J^\mu = 0$$

$$J^\mu = \bar{\psi} \gamma^\mu \psi$$

⇝ CONSERVED CHARGE (q : ELECTRIC CHARGE OF FERMION)
SPIN 1/2

$$Q = q \int d^3\bar{x} \quad J^0 (x)$$

$$= q \int d^3\bar{x} \quad \psi^+(x) \, \psi(x)$$

• SECOND QUANTIZATION OF DIRAC FIELD

↳ TREAT $\psi(x)$ AND $\overline{\psi}(x)$ AS FIELD OPERATORS WHICH CAN CREATE OR ANNIHILATE A DIRAC PARTICLE AT POSITION x.

⇝ WE LIKE TO MAKE A NORMAL MODE EXPANSION OF $\psi, \overline{\psi}$

EACH MODE CORRESPONDS WITH PARTICLE WITH MOMENTUM $\bar{p}$ & SPIN PROJECTION ALONG $\pm s$ (IN REST FRAME)

$$u(p,s) = A \begin{pmatrix} \chi_s \\ c\dfrac{\bar{\sigma}.\bar{p}}{E_p + m_0 c^2}\chi_s \end{pmatrix} \rightsquigarrow (\not{p} - m_0 c)\, u(p,s) = 0$$

$$v(p,s) = A \begin{pmatrix} c\dfrac{\bar{\sigma}.\bar{p}}{E_p + m_0 c^2}\chi_s' \\ \chi_s' \end{pmatrix} \rightsquigarrow (\not{p} + m_0 c)\, v(p,s) = 0$$

⇝ WE WILL SIMPLIFY NOTATION AND DENOTE SECOND ARGUMENT AS VALUE $(\pm \frac{1}{2})$ OF SPIN PROJECTION (s_z) ALONG AXIS z, i.e. $s_z = \pm \frac{1}{2}$ (IN UNITS $\hbar$)

$$u(p, s_z) \rightsquigarrow \chi_{s_z = +\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \chi_{s_z = -\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v(p, s_z) \rightsquigarrow \quad \chi'_{s_z = +\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \quad \chi'_{s_z = -\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

NOTE : OPPOSITE SPIN LABELING FOR v (ANTI-PARTICLE) AS FOR u (PARTICLE)

$\rightsquigarrow$ NORMALIZATION (A) OF SPINORS CHOSEN SUCH THAT

$$\bar{u}(p, s_z)\, u(p, s_z') = \delta_{s_z s_z'}$$

$$\bar{v}(p, s_z)\, v(p, s_z') = -\delta_{s_z s_z'}$$

CHECK :
$$A^2 \left(\chi^+_{s_z} \quad - \chi^+_{s_z} \frac{c\,\bar{\sigma}.\bar{p}}{E_p + m_0 c^2} \right) \begin{pmatrix} \chi_{s_z'} \\ \frac{c\,\bar{\sigma}.\bar{p}}{E_p + m_0 c^2} \chi_{s_z'} \end{pmatrix}$$

$$= A^2 \chi^+_{s_z} \left(1 - \frac{c^2 (\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p})}{(E_p + m_0 c^2)^2} \right) \chi_{s_z'}$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \left(1 - \frac{c^2 \bar{p}^2}{(E_p + m_0 c^2)^2} \right)$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \left(1 - \frac{E_p^2 - m_0^2 c^4}{(E_p + m_0 c^2)^2} \right)$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \left(1 - \frac{E_p - m_0 c^2}{E_p + m_0 c^2} \right)$$

$$= A^2 \chi^+_{s_z} \chi_{s_z'} \frac{2 m_0 c^2}{E_p + m_0 c^2}$$

$$\Downarrow$$

USING $\chi^{+}_{s_z} \chi_{s_z'} = \delta_{s_z s_z'}$

$$\bar{u}(p, s_z)\, u(p, s_z') = \delta_{s_z s_z'}\, A^2 \frac{2 m_0 c^2}{E_p + m_0 c^2}$$

$$= \delta_{s_z s_z'}$$

$$\Updownarrow$$

$$\boxed{A = \sqrt{\frac{E_p + m_0 c^2}{2 m_0 c^2}}}$$

⇝ CHECK THAT WITH THE ABOVE NORMALIZATION

$$u^{+}(p, s_z)\, u(p, s_z') = \frac{E_p}{m_0 c^2}\, \delta_{s_z s_z'}$$

$$v^{+}(p, s_z)\, v(p, s_z') = \frac{E_p}{m_0 c^2}\, \delta_{s_z s_z'}$$

⇝ NORMAL MODES ⇝ FREE PROPAGATING DIRAC PARTICLE

PLANE WAVE (IN VOLUME V) $\dfrac{e^{-\frac{i}{\hbar} p \cdot x}}{\sqrt{V}}$

NORMAL MODES $\psi^{+}(x) \sim \dfrac{e^{-\frac{i}{\hbar} p \cdot x}}{\sqrt{V}}\, u(p, s_z)$ POS. ENERGY SOLUTION

OR

$\psi^{-}(x) \sim \dfrac{e^{+\frac{i}{\hbar} p \cdot x}}{\sqrt{V}}\, v(p, s_z)$ NEG. ENERGY SOLUTION

$$\left(i\hbar\,\gamma^\mu \partial_\mu - m_0 c\right)\psi^+ = 0 \iff (\not{p} - m_0 c)\,u = 0$$

$$\left(i\hbar\,\gamma^\mu \partial_\mu - m_0 c\right)\psi^- = 0 \iff (\not{p} + m_0 c)\,v = 0$$

↳ EXPANSIONS OF DIRAC FIELDS ψ & $\bar{\psi}$

$$\psi(x) = \sum_{\bar{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V}\right)^{1/2} \left\{ b(\bar{p}, s_z)\, u(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} p\cdot x} + d^+(\bar{p}, s_z)\, v(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} p\cdot x} \right\}$$

$$\bar{\psi}(x) = \sum_{\bar{p}} \sum_{s_z} \left(\frac{m_0 c^2}{E_p V}\right)^{1/2} \left\{ b^+(\bar{p}, s_z)\, \bar{u}(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} p\cdot x} + d(\bar{p}, s_z)\, \bar{v}(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} p\cdot x} \right\}$$

$b(\bar{p}, s_z)$ & $d^+(\bar{p}, s_z)$ ARE EXPANSION COEFFICIENTS WHICH WILL BECOME OPERATORS UPON SECOND QUANTIZATION

NOTE : NORMALIZATION FACTOR $\left(\frac{m_0 c^2}{E_p}\right)^{1/2}$ IS INTRODUCED TO GET SIMPLE ANTI-COMMUTATORS FOR b, d AFTER SECOND QUANTIZATION

SECOND QUANTIZATION

IMPOSE ANTI-COMMUTATION RELATIONS FOR EXPANSION COEFFICIENTS b & d

$$\{ b(\bar{p}, s_z), b^+(\bar{p}', s_z') \} = \delta_{\bar{p}\bar{p}'} \delta_{s_z s_z'}$$

$$\{ d(\bar{p}, s_z), d^+(\bar{p}', s_z') \} = \delta_{\bar{p}\bar{p}'} \delta_{s_z s_z'}$$

$$\{ b(\bar{p}, s_z), b(\bar{p}', s_z') \} = 0$$

$$\{ d(\bar{p}, s_z), d(\bar{p}', s_z') \} = 0$$

$$\{ b(\bar{p}, s_z), d(\bar{p}', s') \} = \{ b(\bar{p}, s_z), d^+(\bar{p}', s_z') \} = 0$$

$$\{ d(\bar{p}, s_z), b(\bar{p}', s_z') \} = \{ d(\bar{p}, s_z), b^+(\bar{p}', s_z') \} = 0$$

$\left\{ \begin{matrix} b(\bar{p}, s_z) \\ b^+(\bar{p}, s_z) \end{matrix} \right\}$ IS INTERPRETED AS $\left\{ \begin{matrix} \text{ANNIHILATION} \\ \text{CREATION} \end{matrix} \right\}$ OPERATOR OF DIRAC PARTICLE WITH MOMENTUM $\bar{p}$ & SPIN PROJ. s_z

$\left\{ \begin{matrix} d(\bar{p}, s_z) \\ d^+(\bar{p}, s_z) \end{matrix} \right\}$ IS INTERPRETED AS $\left\{ \begin{matrix} \text{ANNIHILATION} \\ \text{CREATION} \end{matrix} \right\}$ OPERATOR OF DIRAC ANTI-PARTICLE WITH MOMENTUM $\bar{p}$ & SPIN PROJ. s_z

⇝ VACUUM $|0\rangle$

$$b(\bar{p}, s_z)\,|0\rangle = 0$$

$$d(\bar{p}, s_z)\,|0\rangle = 0$$

⇝ NUMBER OPERATORS

$b^+(\bar{p}, s_z)\, b(\bar{p}, s_z)$: # PARTICLES WITH $\bar{p}, s_z$

$d^+(\bar{p}, s_z)\, d(\bar{p}, s_z)$: # ANTI-PARTICLES WITH $\bar{p}, s_z$

⇝ ANTI-COMMUTATION RELATION FOR FIELD OPERATORS

FROM ANTI-COMMUTATION RELATIONS FOR b, d

AT <u>EQUAL TIME</u> t !

$$\left\{ \begin{array}{l} x^\mu = (ct, \bar{x}) \\ x'^\mu = (ct, \bar{x}') \end{array} \right.$$

$$\left\{ \psi_\alpha(\bar{x}, t), \psi_\beta^+(\bar{x}', t) \right\}$$

$$= \sum_{\bar{p} s_z} \sum_{\bar{p}' s_z'} \frac{m_0 c^2}{V (E_p E_{p'})^{1/2}} \cdot \Big\{ b(\bar{p}, s_z)\, u_\alpha(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} p.x} + d^+(\bar{p}, s_z)\, v_\alpha(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} p.x},$$

$$b^+(\bar{p}', s_z')\, u_\beta^+(\bar{p}', s_z')\, e^{+\frac{i}{\hbar} p'.x'} + d(\bar{p}', s_z')\, v_\beta^+(\bar{p}', s_z')\, e^{-\frac{i}{\hbar} p'.x'} \Big\}$$

$$= \sum_{\bar{p}} \sum_{s_z} \frac{m_0 c^2}{E_p V} \Big(u_\alpha(\bar{p}, s_z)\, u_\beta^+(\bar{p}, s_z)\, e^{+\frac{i}{\hbar} \bar{p}.(\bar{x} - \bar{x}')}$$

$$+ v_\alpha(\bar{p}, s_z)\, v_\beta^+(\bar{p}, s_z)\, e^{-\frac{i}{\hbar} \bar{p}.(\bar{x} - \bar{x}')} \Big)$$

$$\left\{ \begin{aligned} u^{+} &= \bar{u}\,\gamma^{0} \\ v^{+} &= \bar{v}\,\gamma^{0} \end{aligned} \right.$$

$$\sum_{s_z} u(\bar{p}, s_z)\, u^{+}(\bar{p}, s_z) = \underbrace{\sum_{s_z} u(\bar{p}, s_z)\, \bar{u}(\bar{p}, s_z)}_{\frac{(\not{p} + m_0 c)}{2 m_0 c}}\, \gamma^{0} \quad \text{CHECK !}$$

$$\sum_{s_z} v(\bar{p}, s_z)\, v^{+}(\bar{p}, s_z) = \underbrace{\sum_{s_z} v(\bar{p}, s_z)\, \bar{v}(\bar{p}, s_z)}_{-\frac{(-\not{p} + m_0 c)}{2 m_0 c}}\, \gamma^{0} \quad \text{CHECK !}$$

$$\therefore \quad \left\{ \psi_{\alpha}(\bar{x}, t),\, \psi_{\beta}^{+}(\bar{x}', t) \right\}$$

$$= \sum_{\bar{p}} \frac{m_0 c^2}{E_p V} \left(\left[\frac{(\not{p} + m_0 c)}{2 m_0 c} \gamma^{0} \right]_{\alpha\beta} e^{\frac{i}{\hbar} \bar{p}\cdot(\bar{x} - \bar{x}')} + \left[\left(\frac{\not{p} - m_0 c}{2 m_0 c} \right) \gamma^{0} \right]_{\alpha\beta} e^{-\frac{i}{\hbar} \bar{p}\cdot(\bar{x} - \bar{x}')} \right)$$

$$\downarrow \quad \sum_{\bar{p}} e^{\frac{i}{\hbar} \bar{p}\cdot(\bar{x} - \bar{x}')} = \frac{V}{(2\pi)^3 \hbar^3} \int d^3\bar{p}\; e^{\frac{i}{\hbar} \bar{p}\cdot(\bar{x} - \bar{x}')}$$

$$= V\, \delta^{3}(\bar{x} - \bar{x}')$$

ONLY p^0 TERM SURVIVES $\qquad p^0 = E_p / c$

$$= \frac{m_0 c^2}{E_p}\, \delta^{3}(\bar{x} - \bar{x}')\, \frac{2 E_p / c}{2 m_0 c}\, \delta_{\alpha\beta}$$

$$\therefore \quad \underline{\underline{\left\{ \psi_{\alpha}(\bar{x}, t),\, \psi_{\beta}^{+}(\bar{x}', t) \right\} = \delta_{\alpha\beta}\, \delta^{3}(\bar{x} - \bar{x}')}}$$

ANTI-COMMUTATION RELATIONS FOR FIELDS

ANALOGOUSLY

$$\left\{ \psi_\alpha(\bar{x},t),\ \psi_\beta(\bar{x}',t) \right\} = 0$$

$$\left\{ \psi_\alpha^+(\bar{x},t),\ \psi_\beta^+(\bar{x}',t) \right\} = 0$$

↳ NORMAL ORDERING

WHEN CALCULATING PHYSICAL QUANTITIES, SUCH AS ENERGY, MOMENTUM, CHARGE,... WE WANT TO EXPRESS THEM RELATIVE TO VACUUM.

CAN BE ACHIEVED BY 'NORMAL ORDERING' (N) OF OPERATORS

↓

IN NORMAL ORDERED PRODUCT : ALL CREATION OPERATORS STAND TO LEFT OF ANNIHILATION OPERATORS

e.g.
$$N(b^+ b) = b^+ b$$
$$N(b\, b^+) = -b^+ b$$
$$N(b\, d^+) = -d^+ b$$
$$\vdots$$

i.e. ANY CONSTANT TERM (VACUUM EXPECTATION VALUE) GETS SUBTRACTED

⇓

∴ VACUUM EXPECTATION VALUE OF NORMAL ORDERED PRODUCT IS ZERO

$$N(A.B) = A.B - \langle 0 | A.B | 0 \rangle$$

↳ CHARGE IN 2° QUANTIZATION

$$Q = \int d^3\bar{x} \quad N\left(\psi^+(\bar{x},t)\,\psi(\bar{x},t)\right)$$

$$= \sum_{\bar{p}} \sum_{\bar{p}'} \sum_{s_z} \sum_{s_z'} \frac{m_0 c^2}{V} \frac{1}{\sqrt{E_p E_{p'}}}$$

$$\cdot \int d^3\bar{x} \left\{ b^+(\bar{p}\, s_z)\, b(\bar{p}' s_z')\, u^+(\bar{p}\, s_z)\, u(\bar{p}' s_z')\, e^{\frac{i}{\hbar}(p-p')\cdot x} \right.$$

$$+ b^+(\bar{p}\, s_z)\, d(\bar{p}' s_z')\, u^+(\bar{p}\, s_z)\, v(\bar{p}' s_z')\, e^{\frac{i}{\hbar}(p+p')\cdot x}$$

$$+ d(\bar{p}\, s_z)\, b(\bar{p}' s_z')\, v^+(\bar{p}\, s_z)\, u(\bar{p}' s_z')\, e^{-\frac{i}{\hbar}(p+p')\cdot x}$$

$$\left. - d^+(\bar{p}' s_z')\, d(\bar{p}\, s_z)\, v^+(\bar{p}\, s_z)\, v(\bar{p}' s_z')\, e^{-\frac{i}{\hbar}(p-p')\cdot x} \right\}$$

DUE TO NORMAL ORDERING

$$\frac{1}{V} \int d^3\bar{x}\; e^{\frac{i}{\hbar}(p-p')\cdot x}$$

$$= \frac{1}{V} \int d^3\bar{x}\; e^{\frac{i}{\hbar}(E_p - E_p')t} \cdot e^{-\frac{i}{\hbar}(\bar{p}-\bar{p}')\cdot \bar{x}}$$

$$= \delta_{\bar{p}\bar{p}'}$$

$$= \sum_{\bar{p}} \sum_{s_z} \sum_{s_z'} \frac{m_0 c^2}{E_p}$$

$$\cdot \left\{ b^+(\bar{p}\, s_z)\, b(\bar{p}\, s_z')\, u^+(\bar{p}\, s_z)\, u(\bar{p}\, s_z') \right.$$

$$+ b^+(\bar{p}\, s_z)\, d(-\bar{p}\, s_z')\, u^+(\bar{p}\, s_z)\, v(-\bar{p}\, s_z')$$

$$+ \; d(\bar{p}\, s_z) \; b(-\bar{p}\, s_z') \; v^+(\bar{p}\, s_z) \; u(-\bar{p}\, s_z')$$

$$- \; d^+(\bar{p}\, s_z') \; d(\bar{p}\, s_z) \; v^+(\bar{p}\, s_z) \; v(\bar{p}\, s_z') \Big\}$$

$$u^+(\bar{p}\, s_z) \; u(\bar{p}\, s_z') = \frac{E_p}{m_0 c^2} \, \delta_{s_z s_z'}$$

$$v^+(\bar{p}\, s_z) \; v(\bar{p}\, s_z') = \frac{E_p}{m_0 c^2} \, \delta_{s_z s_z'}$$

$$v^+(\bar{p}\, s_z) \; u(-\bar{p}\, s_z') = 0$$

$$u^+(\bar{p}\, s_z) \; v(-\bar{p}\, s_z') = 0$$

$$\therefore \quad \boxed{Q = \sum_{\bar{p}} \sum_{s_z} \Big\{ b^+(\bar{p}\, s_z)\, b(\bar{p}\, s_z) - d^+(\bar{p}\, s_z)\, d(\bar{p}\, s_z) \Big\}}$$

$b^+ b$: # PARTICLES

$d^+ d$: # ANTI - PARTICLES

<u>NOTE</u> : ANTI - PARTICLES HAVE OPPOSITE CHARGE AS PARTICLES.

ELECTRICAL CHARGE IS $(-e)\, Q$

$\hookrightarrow -e$: CHARGE OF e^- (WITH $e > 0$)

$+e$ IS CHARGE OF POSITRON e^+

ENERGY / MOMENTUM IN 2° QUANTIZATION

ANALOGOUSLY ONE CAN SHOW THAT

$$H = \sum_{\bar{p}} \sum_{s_z} E_p \left(b^+(\bar{p}\, s_z)\, b(\bar{p}\, s_z) + d^+(\bar{p}\, s_z)\, d(\bar{p}\, s_z) \right)$$

$$\bar{P} = \sum_{\bar{p}} \sum_{s_z} \bar{p} \left(b^+(\bar{p}\, s_z)\, b(\bar{p}\, s_z) + d^+(\bar{p}\, s_z)\, d(\bar{p}\, s_z) \right)$$