IV

RELATIVISTIC FERMIONS

1) DIRAC EQUATION

- DERIVATION OF DIRAC EQ.

↳ SCHRÖDINGER EQ.

$$\hat{H}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

↳ DIRAC (1928) WAS LOOKING FOR A RELATIVISTIC COVARIANT VERSION OF SCHRÖDINGER EQ. LINEAR IN SPACE-TIME DERIVATIVES

$$\left[-i\hbar c\left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m_0 c^2\right]\Psi$$

$$= i\hbar \frac{\partial \Psi}{\partial t} \qquad = c\hat{\vec{\alpha}} \cdot \hat{\vec{p}} + \hat{\beta} m_0 c^2$$

WITH $\hat{\vec{p}} = -i\hbar \vec{\nabla}$

$\hat{\alpha}_i$ CANNOT BE NUMBERS AS EQ. SHOULD BE INVARIANT UNDER SPATIAL ROTATIONS

↓

TRY: $\hat{\alpha}_i$ ARE $N \times N$ MATRICES, Ψ : $N \times 1$ COLUMN MATRIX

$$\Psi = \begin{bmatrix} \psi_1 \\ \vdots \\ \psi_N \end{bmatrix}$$

$$\underbrace{\left[-i\hbar c\left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m_0 c^2\right]_{\sigma\tau}}_{\hat{H}} \psi_\tau$$

$$= i\hbar \frac{\partial \psi_\sigma}{\partial t}$$

SUM OVER $\tau = 1 \dots N$ (SUMMATION CONVENTION)

NOTE : σ, τ ARE USUAL MATRIX INDICES !
WE DO NOT NEED TO MAKE DISTINCTION
BETWEEN COVARIANT / CONTRAVARIANT
AS IN CASE OF FOUR-VECTORS

↳ WE LIKE TO RECOVER RELATIVISTIC ENERGY-MOMENTUM RELATION FOR FREE PARTICLE

⇓

POSSIBLE IF EACH COMPONENT ψ_σ SATISFIES KLEIN-GORDON EQUATION

⇝ REQUIRE $\left(\partial_\mu \partial^\mu + \frac{m_0^2 c^2}{\hbar^2}\right) \psi_\sigma = 0$

PROOF

$$i\hbar \frac{\partial \psi_\sigma}{\partial t} = \left[-i\hbar c\left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m_0 c^2\right]_{\sigma\tau} \psi_\tau$$

$$\Downarrow \quad i\hbar \frac{\partial}{\partial t}$$

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left[-i\hbar c\left(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}\right) + \hat{\beta} m_0 c^2\right]_{\sigma\tau} i\hbar \frac{\partial \psi_\tau}{\partial t}$$

N × N MATRIX EQ.

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = \left[-i\hbar c \sum_{i=1}^{3} \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0 c^2\right]$$

$$\cdot \left[-i\hbar c \sum_{j=1}^{3} \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0 c^2\right] \Psi$$

$$= -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \Psi}{\partial x^i \partial x^j}$$

$$- i\hbar m_0 c^3 \sum_{i=1}^{3} \left(\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i\right) \frac{\partial \Psi}{\partial x^i}$$

$$+ (m_0 c^2)^2 \hat{\beta}^2 \Psi$$

$\Downarrow$

WE WANT THIS TO BE IN FORM

$$\frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} - \nabla^2 \Psi + \frac{m_0^2 c^2}{\hbar^2} \Psi = 0$$

$\Downarrow$

REQUIRE:

$$\boxed{\begin{aligned} \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2\delta_{ij} \mathbb{1}_{N\times N} \\ \hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= 0 \\ \hat{\alpha}_i^2 = \hat{\beta}^2 &= \mathbb{1}_{N\times N} \end{aligned}}$$

↳ IN ADDITION : WE WANT $\hat{H}$ TO BE HERMITIAN

$\Downarrow$

$\hat{\alpha}_i, \hat{\beta}$ HAVE TO BE HERMITIAN

$$\boxed{\begin{aligned} \hat{\alpha}_i^+ &= \hat{\alpha}_i \\ \hat{\beta}^+ &= \hat{\beta} \end{aligned}}$$

(REAL EIGENVALUES)

- $\hat{\alpha}_i^2 = \hat{\beta}^2 = 1 \quad \Rightarrow$ EIGENVALUES ARE ± 1

- $\hat{\alpha}_i = -\hat{\beta}\,\hat{\alpha}_i\,\hat{\beta}$

$$\Downarrow$$

$$\mathrm{Tr}\,\hat{\alpha}_i = -\mathrm{Tr}\left\{\hat{\beta}\,\hat{\alpha}_i\,\hat{\beta}\right\} = -\mathrm{Tr}\left\{\hat{\alpha}_i\right\}$$

$$\Downarrow$$

$$\left.\begin{array}{l} \mathrm{Tr}\,\hat{\alpha}_i = 0 \\ \text{ANALOGOUSLY} \quad \mathrm{Tr}\,\hat{\beta} = 0 \end{array}\right\} \text{TRACELESS.}$$

- FOR $N = 2 \Rightarrow$ ONLY 3 ANTI-COMMUTING MATRICES POSSIBLE (PAULI-MATRICES)

- SINCE $\hat{\alpha}_i, \hat{\beta}$ HAVE EIGENVALUES ± 1 & ARE TRACELESS THEY MUST HAVE SAME NUMBER OF $+1$ AS -1 EIGENVALUES $\Rightarrow$ ONLY <u>EVEN</u> N ARE POSSIBLE

↳ SMALLEST POSSIBLE DIMENSION: $\underline{N = 4}$

$\hat{\alpha}_i , \hat{\beta}$ DIRAC MATRICES : (4 × 4 MATRICES)

⇓

DIRAC REPRESENTATION

$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix}$	$\hat{\beta}_i = \begin{pmatrix} \mathbb{1}_{2\times 2} & 0 \\ 0 & -\mathbb{1}_{2\times 2} \end{pmatrix}$

WITH $\hat{\sigma}_i$ PAULI-MATRICES (2 × 2 MATRICES)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

NOTE $\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2 \delta_{ij} \mathbb{1}_{2\times 2}$

⇝ $\hat{\alpha}_i , \hat{\beta}$ TRACELESS & HERMITIAN

⇝ CHECK OF ANTI-COMMUTATION RELATIONS ✓

$$\hat{\alpha}_i^2 = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{2\times 2} & \\ & \mathbb{1}_{2\times 2} \end{pmatrix} \stackrel{!}{=} \mathbb{1}_{4\times 4}$$

$$\hat{\beta}^2 \stackrel{!}{=} \mathbb{1}_{4\times 4}$$

$$\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} \stackrel{!}{=} 0$$

$$\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i = \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix}$$

$$\stackrel{!}{=} 2 \delta_{ij} \cdot \mathbb{1}_{4\times 4}$$

⇝ WE CAN CHOOSE ANOTHER REPRESENTATION OF $\hat{\alpha}_i$, $\hat{\beta}$ BY <u>UNITARY</u> TF.

e.g. $\hat{\alpha}_i' = U \hat{\alpha}_i U^{-1}$

$$\hat{\beta}' = U \hat{\beta} U^{-1}$$

$$\hat{\alpha}_i' \hat{\alpha}_j' + \hat{\alpha}_j' \hat{\alpha}_i' = U \hat{\alpha}_i U^{-1} U \hat{\alpha}_j U^{-1} + U \hat{\alpha}_j U^{-1} U \hat{\alpha}_i U^{-1}$$

$$= U \underbrace{\left(\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i \right)}_{2\delta_{ij} \mathbb{1}_{4\times 4}} U^{-1}$$

$$= 2\delta_{ij} \underbrace{U U^{-1}}_{\mathbb{1}_{4\times 4}}$$

⇓

PHYSICS DOES <u>NOT</u> DEPEND ON SPECIFIC REPRESENTATION OF DIRAC MATRICES $\hat{\alpha}_i$, $\hat{\beta}$

↳ ABOVE DIRAC REPRESENTATION IS OFTEN CONVENIENT AND USED IN MANY CASES

∴ DIRAC EQUATION (4×1 MATRIX EQ.)

$$\boxed{\left[-i\hbar c\, \hat{\vec{\alpha}} \cdot \vec{\nabla} + \hat{\beta} m_0 c^2 \right] \Psi = i\hbar \frac{\partial \Psi}{\partial t}}$$

$\hat{\vec{\alpha}} \equiv (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ EACH $\hat{\alpha}_i$: 4×4 MATRIX

- CONTINUITY EQUATION

WE LIKE TO HAVE AN EQ. FOR 'PROBABILITY DENSITY'

$\Psi^+ \Psi$

⇝ DIRAC EQ.

$$-i\hbar c\,\hat{\alpha}_i \frac{\partial \Psi}{\partial x^i} + \hat{\beta}\, m_0 c^2 \Psi = i\hbar \frac{\partial \Psi}{\partial t} \qquad (*)$$

⇝ HERMITION CONJUGATE EQ.

$$+i\hbar c \frac{\partial \Psi^+}{\partial x^i}\hat{\alpha}_i + \Psi^+ \hat{\beta}\, m_0 c^2 = -i\hbar \frac{\partial \Psi^+}{\partial t} \qquad (**)$$

⇝ $\Psi^+ (*) - (**)\Psi$

$$i\hbar \left(\Psi^+ \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^+}{\partial t}\Psi \right)$$

$$= -i\hbar c\left(\Psi^+ \hat{\alpha}_i \frac{\partial \Psi}{\partial x^i} + \frac{\partial \Psi^+}{\partial x^i}\hat{\alpha}_i \Psi \right)$$

$$+ m_0 c^2 \cancel{\left(\Psi^+ \hat{\beta} \Psi - \Psi^+ \hat{\beta} \Psi \right)}$$

$\Downarrow$

$$\frac{\partial \rho}{\partial t} + \overline{\nabla} . \overline{J} = 0 \qquad \text{CONTINUITY EQ.}$$

$\rho = \Psi^+ \Psi$	$\overline{J} = c\, \Psi^+ \hat{\alpha} \Psi$

ρ : PROBABILITY DENSITY

(POSITIVE DEFINITE)

$\bar{J}$: PROBABILITY CURRENT $\bar{J} = (J^1, J^2, J^3)$

$$J^i \equiv c\,\Psi^+ \hat{\alpha}^i \Psi$$

- SOLUTIONS OF DIRAC EQ. FOR FREE PARTICLE

$$\Psi(\bar{x}, t) = \psi(\bar{x})\, e^{-\frac{i}{\hbar}Et}$$

$$\left[-i\hbar c\, \hat{\bar{\alpha}} \cdot \bar{\nabla} + \hat{\beta} m_0 c^2\right] \psi(\bar{x}) = E\, \psi(\bar{x})$$

$\psi(\bar{x})$ IS 4 x 1 COLUMN MATRIX

↳ SPINOR

$$\psi(\bar{x}) = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$ WITH φ & χ 2 x 1 COLUMN MATRICES

$$\left[-i\hbar c \begin{pmatrix} 0 & \bar{\sigma}\cdot\bar{\nabla} \\ \bar{\sigma}\cdot\bar{\nabla} & 0 \end{pmatrix} + m_0 c^2 \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}\right] \begin{bmatrix} \varphi \\ \chi \end{bmatrix} = E \begin{bmatrix} \varphi \\ \chi \end{bmatrix}$$

$$\begin{cases} -i\hbar c\, \bar{\sigma}\cdot\bar{\nabla}\, \chi + m_0 c^2 \varphi = E \varphi \\ -i\hbar c\, \bar{\sigma}\cdot\bar{\nabla}\, \varphi - m_0 c^2 \chi = E \chi \end{cases}$$

SOLUTIONS WITH GOOD MOMENTUM

$$\psi(\bar{x}) = \begin{pmatrix} \varphi_0 \\ \chi_0 \end{pmatrix} e^{+\frac{i}{\hbar}\bar{p}.\bar{x}}$$

WITH φ_0, χ_0 INDEP. OF $\bar{x}$

$$\begin{cases} c\,\bar{\sigma}.\bar{p}\,\chi_0 + m_0 c^2 \varphi_0 = E\varphi_0 \\ c\,\bar{\sigma}.\bar{p}\,\varphi_0 - m_0 c^2 \chi_0 = E\chi_0 \end{cases}$$

2° EQ: $$\chi_0 = \frac{1}{E + m_0 c^2}\, c\,\bar{\sigma}.\bar{p}\,\varphi_0$$

$\downarrow$

IN 1° EQ. $$c\,\bar{\sigma}.\bar{p}\,\chi_0 = (E - m_0 c^2)\varphi_0$$

$$\Downarrow$$

$$\frac{c^2}{E + m_0 c^2}(\bar{\sigma}.\bar{p})(\bar{\sigma}.\bar{p})\varphi_0 = (E - m_0 c^2)\varphi_0$$

$$\Downarrow$$

$$\frac{c^2}{E + m_0 c^2}\bar{p}^2 = (E - m_0 c^2)$$

$$c^2\bar{p}^2 = E^2 - (m_0 c^2)^2$$

$$\therefore \quad E^2 = c^2 p^2 + m_0^2 c^4$$

RELATIVISTIC ENERGY - MOMENTUM RELATION

SOLUTIONS OF POSITIVE & NEGATIVE ENERGY

$$E = \lambda E_{\bar{p}}$$

WITH $\lambda = \pm 1$ $\qquad E_{\bar{p}} = + \sqrt{c^2 \bar{p}^2 + m_0^2 c^4}$

$$\boxed{\Psi_{\bar{p}\lambda}(\bar{x},t) = N \, e^{-\frac{i}{\hbar}\left(\lambda E_{\bar{p}} t - \bar{p}\cdot\bar{x}\right)} \begin{pmatrix} \varphi_0 \\ \dfrac{c\,\bar{\sigma}\cdot\bar{p}}{\lambda E_{\bar{p}} + m_0 c^2}\,\varphi_0 \end{pmatrix}}$$

WITH N : NORMALIZATION FACTOR

↳ NORMALIZATION

FROM CONDITION

$$\int d^3\bar{x} \; \Psi^{+}_{\bar{p}\lambda}(\bar{x},t)\, \Psi_{\bar{p}'\lambda'}(\bar{x},t) = \delta_{\lambda\lambda'} \, (2\pi)^3 \hbar^3 \, \delta^3(\bar{p} - \bar{p}')$$

$$N^2 \begin{pmatrix} \varphi_0^+ & \varphi_0^+ \dfrac{c\,\bar{\sigma}\cdot\bar{p}}{\lambda E_{\bar{p}} + m_0 c^2} \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \dfrac{c\,\bar{\sigma}\cdot\bar{p}}{\lambda E_{\bar{p}} + m_0 c^2}\,\varphi_0 \end{pmatrix} = 1$$

$$N^2 \left(\varphi_0^+ \varphi_0 + \varphi_0^+ \frac{c^2\, \bar{p}^2}{(\lambda E_{\bar{p}} + m_0 c^2)^2}\, \varphi_0 \right) = 1$$

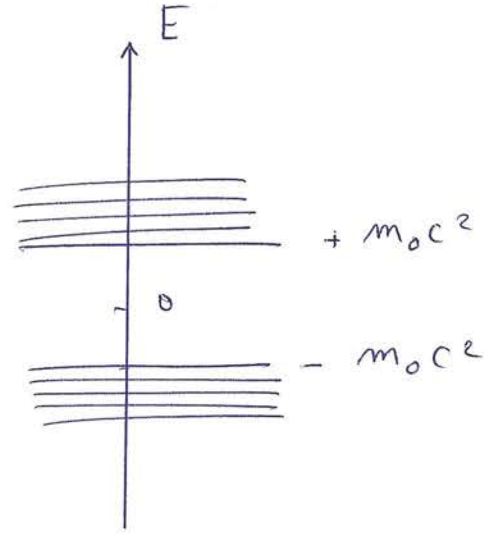
$\downarrow$ FOR $\underline{\varphi_0^+ \varphi_0 = 1}$

$$N^2 \left(1 + \frac{(\lambda E_{\bar{p}} + m_0 c^2)(\lambda E_{\bar{p}} - m_0 c^2)}{(\lambda E_{\bar{p}} + m_0 c^2)^2} \right) = 1$$

$$N^2 \cdot \left(1 + \frac{\lambda E_{\bar{p}} - m_0 c^2}{\lambda E_{\bar{p}} + m_0 c^2} \right) = 1$$

$$\boxed{N = \sqrt{\frac{\lambda E_{\bar{p}} + m_0 c^2}{2 \lambda E_{\bar{p}}}}}$$

↳ SPECTRUM



FOR EVERY MOMENTUM $\bar{p}$ THERE ARE 2 ENERGY STATES

$E = \lambda E_{\bar{p}}$ WITH $\lambda = \pm 1$

- LOCAL PHASE TF. $U(1)$

$$\phi(x) \xrightarrow{U(1)} e^{iq\chi(x)} \phi(x) \equiv \phi'(x)$$

$$\partial_\mu \phi \xrightarrow{U(1)} e^{iq\chi(x)} \Big[\partial_\mu \phi + \underbrace{iq\, \phi\, (\partial_\mu \chi)}_{\text{EXTRA TERM}} \Big]$$

- TO PRESERVE INV. UNDER $U(1)$ LOCAL PHASE TF

$$\partial_\mu \phi \underset{\text{REPLACE}}{\Rightarrow} D_\mu \phi \equiv (\partial_\mu + iq A_\mu)\, \phi$$

& REQUIRE $D_\mu \phi \xrightarrow{U(1)} e^{iq\chi(x)} D_\mu \phi$

- THIS IS ACHIEVED BY GAUGE TF FOR A_μ FIELD

$$\partial_\mu \phi' + iq A'_\mu \phi' = e^{iq\chi} \left[\partial_\mu \phi + iq A_\mu \phi \right]$$

$$\phi'(x) = e^{iq\chi(x)} \phi(x)$$

$$e^{iq\chi} \left[\partial_\mu \phi + iq\, \phi\, (\partial_\mu \chi) + iq A'_\mu \phi \right]$$

$$= e^{iq\chi} \left[\partial_\mu \phi + iq A_\mu \phi \right]$$

$$\therefore \quad A'_\mu = A_\mu - \partial_\mu \chi$$

$$\left.\begin{aligned} \bar{B} &= \bar{\nabla} \times \bar{A} \\ \bar{E} &= -\bar{\nabla} A^0 - \frac{\partial \bar{A}}{\partial t} \end{aligned}\right\} \text{ STAY INVARIANT}$$

↳ HELICITY

DIRAC EQ.

$$\underbrace{\left(c\,\hat{\bar{\alpha}}\cdot\hat{\bar{p}} + \hat{\beta}\, m_0 c^2 \right)}_{\hat{H}} \Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

WITH $\hat{\bar{p}} = -i\hbar \bar{\nabla}$

INTRODUCE OPERATOR $\hat{\bar{\Sigma}}\cdot\hat{\bar{p}} \equiv \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix} \cdot \hat{\bar{p}}$

↓

$\hat{\bar{\Sigma}}\cdot\hat{\bar{p}}$ COMMUTES WITH $\hat{H}$

$$\left[\hat{H}, \hat{\bar{\Sigma}}\cdot\hat{\bar{p}} \right] = \left[c\,\hat{\bar{\alpha}}\cdot\hat{\bar{p}} + \hat{\beta}\, m_0 c^2 , \hat{\bar{\Sigma}}\cdot\hat{\bar{p}} \right]$$

$$= c \left[\hat{\bar{\alpha}}\cdot\hat{\bar{p}} , \hat{\bar{\Sigma}}\cdot\hat{\bar{p}} \right]$$

$$= c \left\{ \begin{pmatrix} 0 & \bar{\sigma}\cdot\hat{\bar{p}} \\ \bar{\sigma}\cdot\hat{\bar{p}} & 0 \end{pmatrix} \begin{pmatrix} \bar{\sigma}\cdot\hat{\bar{p}} & 0 \\ 0 & \bar{\sigma}\cdot\hat{\bar{p}} \end{pmatrix} - \begin{pmatrix} \bar{\sigma}\cdot\hat{\bar{p}} & 0 \\ 0 & \bar{\sigma}\cdot\hat{\bar{p}} \end{pmatrix} \begin{pmatrix} 0 & \bar{\sigma}\cdot\hat{\bar{p}} \\ \bar{\sigma}\cdot\hat{\bar{p}} & 0 \end{pmatrix} \right\}$$

$$= c \left\{ \begin{pmatrix} 0 & (\bar{\sigma}\cdot\hat{\bar{p}})^2 \\ (\bar{\sigma}\cdot\hat{\bar{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\bar{\sigma}\cdot\hat{\bar{p}})^2 \\ (\bar{\sigma}\cdot\hat{\bar{p}})^2 & 0 \end{pmatrix} \right\}$$

$$= 0$$

⇓

$\hat{H}$ & $\hat{\bar{\Sigma}}\cdot\hat{\bar{p}}$ CAN BE DIAGONALIZED SIMULTANEOUSLY

SOLUTIONS OF DIRAC EQ. ARE EIGENSTATES OF $\hat{\bar{\Sigma}} \cdot \hat{\bar{p}}$

$\rightsquigarrow$ INTRODUCE 'SPIN-VECTOR OPERATOR' (4 × 4 MATRIX)

$$\hat{\bar{S}} \equiv \frac{\hbar}{2} \hat{\bar{\Sigma}} = \frac{\hbar}{2} \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$$

$\rightsquigarrow$ HELICITY OPERATOR $\hat{\Lambda}_s$

$$\hat{\Lambda}_s \equiv \hat{\bar{S}} \cdot \frac{\hat{\bar{p}}}{|\bar{p}|}$$

PROJECTION OF SPIN ON MOMENTUM OF PARTICLE

e.g. CONSIDER FREE PARTICLE TRAVELLING ALONG Z-AXIS

$$\bar{p} = (0, 0, p)$$

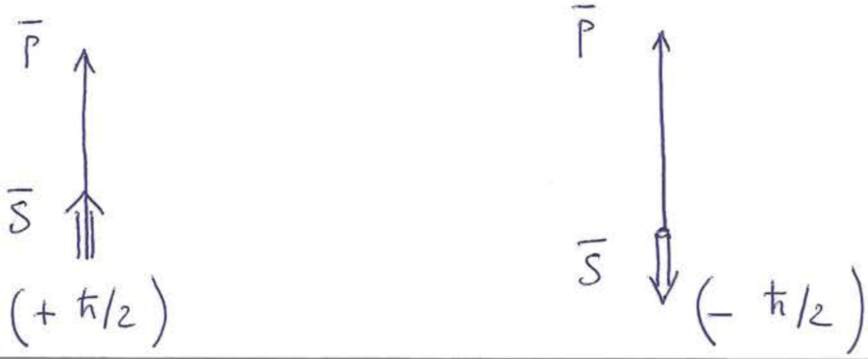
$$\hat{\Lambda}_s = \hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}$$

$\Downarrow$

EIGENSTATES: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$\downarrow$

EIGENVALUES: $+\frac{\hbar}{2}$ $-\frac{\hbar}{2}$ $+\frac{\hbar}{2}$ $-\frac{\hbar}{2}$



∴ SOLUTIONS OF FREE DIRAC EQ. HAVE GOOD HELICITY / SPIN PROJECTION

$$\hat{\bar{S}} \cdot \frac{\hat{\bar{p}}}{|\bar{p}|} \Psi_{\bar{p}\lambda}(\bar{x},t) = \pm \frac{\hbar}{2} \Psi_{\bar{p}\lambda}(\bar{x},t)$$

$$= \hat{\bar{S}} \cdot \frac{\bar{p}}{|\bar{p}|} N e^{-\frac{i}{\hbar}(\lambda E_{\bar{p}} t - \bar{p}.\bar{x})} \begin{bmatrix} \varphi_0 \\ \frac{c\,\bar{\sigma}.\bar{p}}{\lambda E_{\bar{p}} + m_0 c^2} \varphi_0 \end{bmatrix}$$

$$= \frac{\hbar}{2} N e^{-\frac{i}{\hbar}(\lambda E_{\bar{p}} t - \bar{p}.\bar{x})} \begin{bmatrix} \bar{\sigma}.\frac{\bar{p}}{|\bar{p}|} \varphi_0 \\ \frac{c}{|\bar{p}|} \frac{(\bar{\sigma}.\bar{p})^2}{\lambda E_{\bar{p}} + m_0 c^2} \varphi_0 \end{bmatrix}$$

$$= \pm \frac{\hbar}{2} \Psi_{\bar{p}\lambda}(\bar{x},t)$$

$$\Updownarrow$$

$$\underline{\underline{\bar{\sigma} \cdot \frac{\bar{p}}{|\bar{p}|} \varphi_0 = \pm \varphi_0}}$$

φ_0 IS PAULI SPINOR (2 × 1 COLUMN MATRIX)

e.g. FOR PARTICLE TRAVELING ALONG Z-AXIS

$$\sigma_z \varphi_0 = \pm \varphi_0 \nearrow \varphi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SPIN PROJECTION ALONG Z-AXIS $(+\hbar/2)$

$$\searrow \varphi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (-\hbar/2)$$

FOR ARBITRARY DIRECTION

$$\hat{p}\ (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\varphi_{s=+\frac{1}{2}} = \begin{bmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{bmatrix}$$

$$\varphi_{s=-\frac{1}{2}} = \begin{bmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{bmatrix}$$

- NON - RELATIVISTIC CORRESPONDENCE OF DIRAC EQ.

⤳ CONSIDER NON-REL. PARTICLE ($v/c \ll 1$) WITH CHARGE e MOVING IN APPLIED ELECTROMAGNETIC FIELD $A^\mu(\Phi, \bar{A})$

$\Downarrow$

DESCRIBED BY 'MINIMAL SUBSTITUTION' (SEE ABOVE)

$$\hat{\bar{p}} \to \hat{\bar{p}} - \frac{e}{c}\bar{A}$$

& ELECTROSTATIC POTENTIAL $e\underline{\Phi}$

$$\underbrace{\left[c\,\hat{\bar{\alpha}}\cdot\left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right) + \hat{\beta} m_0 c^2 + e\underline{\Phi}\right]}_{\hat{H}}\underline{\Psi} = i\hbar\frac{\partial\underline{\Psi}}{\partial t}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = c\,\hat{\bar{\alpha}}\cdot\hat{\bar{p}} + \beta m_0 c^2 \qquad \text{FREE DIRAC HAM.}$$

$$\hat{H}_1 = -e\,\hat{\bar{\alpha}}\cdot\bar{A} + e\underline{\Phi}$$

⤳ NON - REL. REDUCTION OF DIRAC EQ.

$$\underline{\Psi}(\bar{x},t) = e^{-\frac{i}{\hbar}m_0c^2 t}\begin{bmatrix}\varphi \\ \chi\end{bmatrix}$$

$\Uparrow$

* FOR POSITIVE ENERGY SOLUTIONS:

* KINETIC ENERGY APPROXIMATED RELATIVE TO REST MASS

$$i\hbar \frac{\partial \Psi}{\partial t} = (m_o c^2)\,\Psi + e^{-\frac{i}{\hbar} m_o c^2 t} \left(i\hbar \frac{\partial}{\partial t}\right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\Downarrow$$

$$\left(c\,\hat{\alpha}.\hat{\bar{\pi}} + \beta m_o c^2 + e\,\Phi \right) \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = m_o c^2 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} + i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

WITH $\hat{\bar{\pi}} \equiv \hat{\bar{p}} - \frac{e}{c} \bar{A}$

$$\Downarrow$$

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c \begin{pmatrix} \bar{\sigma}.\hat{\bar{\pi}}\;\chi \\ \bar{\sigma}.\hat{\bar{\pi}}\;\varphi \end{pmatrix} - 2 m_o c^2 \begin{pmatrix} 0 \\ \chi \end{pmatrix} + e\,\Phi \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

$$\begin{cases} i\hbar \dfrac{\partial \varphi}{\partial t} = c\,\bar{\sigma}.\hat{\bar{\pi}}\;\chi + e\,\Phi\,\varphi \\[2ex] i\hbar \dfrac{\partial \chi}{\partial t} = c\,\bar{\sigma}.\hat{\bar{\pi}}\;\varphi - 2 m_o c^2 \chi + e\,\Phi\,\chi \end{cases}$$

$\rightsquigarrow$ <u>2^o EQ.</u> : NEGLECT KINETIC ENERGY $i\hbar \frac{\partial \chi}{\partial t}$ CONTRIBUTION
& POTENTIAL ENERGY $e\,\Phi\,\chi$ CONTRIBUTION
RELATIVE TO REST MASS TERM $-2 m_o c^2 \chi$

$$\rightsquigarrow \quad \chi \simeq \frac{\bar{\sigma}.\hat{\bar{\pi}}}{2 m_o c}\,\varphi$$

⇝ <u>1° EQ.</u> :

$$i\hbar \frac{\partial \varphi}{\partial t} \simeq c\left(\bar{\sigma}.\hat{\bar{\pi}}\right) \frac{\bar{\sigma}.\hat{\bar{\pi}}}{2 m_0 c} \varphi + e\Phi \varphi$$

<u>USE</u> $\left(\bar{\sigma}.\hat{\bar{\pi}}\right)\left(\bar{\sigma}.\hat{\bar{\pi}}\right) = \hat{\bar{\pi}}.\hat{\bar{\pi}} + i\,\bar{\sigma}.\left(\hat{\bar{\pi}} \times \hat{\bar{\pi}}\right)$

$$\hat{\bar{\pi}}.\hat{\bar{\pi}} = \left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right).\left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right)$$

$$= \hat{\bar{p}}^2 - \frac{e}{c}\left(\hat{\bar{p}}.\bar{A} + \bar{A}.\hat{\bar{p}}\right) + \frac{e^2}{c^2}\bar{A}^2$$

$$\hat{\bar{\pi}} \times \hat{\bar{\pi}} = \left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right) \times \left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right)$$

$$= -\frac{e}{c}\left(\hat{\bar{p}} \times \bar{A} + \bar{A} \times \hat{\bar{p}}\right)$$

$$= i\hbar \frac{e}{c}\left(\bar{\nabla} \times \bar{A}\right)$$

$$= i\frac{e\hbar}{c}\bar{B}$$

⇓

$$\boxed{i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{\left(\hat{\bar{p}} - \frac{e}{c}\bar{A}\right)^2}{2m_0} - \frac{e\hbar}{2 m_0 c}\bar{\sigma}.\bar{B} + e\Phi\right]\varphi}$$

PAULI EQUATION

↳ DESCRIBES NON-REL. PARTICLE OF SPIN 1/2 MOVING IN AN ELECTROMAGNETIC FIELD

2° TERM : $-\bar{\mu}.\bar{B}$

WITH $\bar{\mu}$: MAGNETIC MOMENT $\bar{\mu} = \frac{e}{2m_0 c} g\bar{S}$

SPIN $\bar{S} = \frac{\hbar}{2}\bar{\sigma}$, $g = 2$: GYROMAGNETIC RATIO

2) LORENTZ COVARIANCE OF DIRAC EQUATION

- COVARIANT FORM

WE WANT TO SHOW THAT UNDER LORENTZ TRANSFORMATION THE DIRAC EQ. KEEPS THE SAME FORM

↳ 4- VECTOR NOTATION

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \overline{\nabla} \right)$$

↳ $$\left[c\,(-i\hbar)\,\hat{\overline{\alpha}}\cdot\overline{\nabla} + \hat{\beta}\, m_0 c^2 \right] \Psi = i\hbar \frac{\partial}{\partial t} \Psi$$

MULTIPLY ON LEFT WITH $\hat{\beta}$

$$i\hbar \left(\hat{\beta}\, \frac{1}{c}\frac{\partial}{\partial t} + \hat{\beta}\,\overline{\alpha}\cdot\overline{\nabla} \right)\Psi - m_0 c\, \Psi = 0$$

⇓

INTRODUCE $\gamma^\mu \equiv (\gamma^0, \overline{\gamma})$

$$= (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$$

WITH
$$\gamma^0 \equiv \hat{\beta}$$
$$\overline{\gamma} \equiv \hat{\beta}\,\hat{\overline{\alpha}}$$

γ^μ ARE 4×4 MATRICES

$$\boxed{\left(i\hbar \quad \gamma^\mu \partial_\mu \ - m_0 c\right) \Psi = 0}$$

OR WITH $\quad \hat{p}_\mu = i\hbar \, \partial_\mu$

$$\boxed{\left(\gamma^\mu \hat{p}_\mu - m_0 c\right) \Psi = 0}$$

∴ DIRAC EQ. IN COVARIANT FORM

$$\hookrightarrow \gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \overline{\gamma} = \begin{pmatrix} 0 & \overline{\sigma} \\ -\overline{\sigma} & 0 \end{pmatrix}$$

$$\boxed{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu} \, \mathbb{1}_{4\times4}}$$

$\leadsto \mu = \nu = 0 \qquad \hat{\beta}^2 = 1 \quad ✓$

$\leadsto \mu = 0, \ \nu = i \qquad \gamma^0 \gamma^i + \gamma^i \gamma^0 = 0$

$$\Updownarrow$$

$$\hat{\beta}^2 \hat{\alpha}_i + \hat{\beta} \hat{\alpha}_i \hat{\beta} = 0$$

$\Updownarrow$ MULTIPLY ON RIGHT WITH $\hat{\beta}$

$$\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i = 0 \quad ✓$$

$\leadsto \mu = i = \nu \qquad 2 \gamma^i \gamma^i = -2$

$$\hat{\beta} \underbrace{\hat{\alpha}_i \hat{\beta}}_{-\hat{\beta}\hat{\alpha}_i} \hat{\alpha}_i = -1$$

$$\Updownarrow$$

$$\hat{\alpha}_i^2 = 1 \quad ✓$$

$\rightsquigarrow \quad \mu = i\,,\ \nu = j \quad (i \neq j)$

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 0$$

$$\Updownarrow$$

$$\hat{\beta}\hat{\alpha}_i\,\hat{\beta}\,\hat{\alpha}_j + \hat{\beta}\hat{\alpha}_j\,\hat{\beta}\,\hat{\alpha}_i = 0$$

$$\Updownarrow$$

$$+\ \hat{\alpha}_i\,\hat{\alpha}_j + \hat{\alpha}_j\,\hat{\alpha}_i = 0 \qquad \checkmark$$

WE FIND BACK COMMUTATION RELATIONS !

<u>NOTE</u> γ^0 IS HERMITIAN

γ^i IS ANTI-HERMITIAN $\left(\gamma^i\right)^\dagger = -\gamma^i$

↳ 'SLASH' NOTATION

IT IS CONVENTIONAL TO INTRODUCE

$$\not{A} \equiv \gamma^\mu A_\mu$$

$$\rightsquigarrow \quad \not{\partial} \equiv \gamma^\mu \partial_\mu$$

$$\rightsquigarrow \quad \hat{\not{p}} \equiv \gamma^\mu \hat{p}_\mu$$

etc

<u>DIRAC EQ.</u> $$\boxed{\left(\hat{\not{p}} - m_0 c\right) \Psi = 0}$$

• LORENTZ COVARIANCE

WE WANT TO DEMONSTRATE THAT DIRAC EQ. REMAINS COVARIANT UNDER A LORENTZ TF.

↳ LORENTZ TF

$$\underline{\underline{x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu}}}$$

$$x'^{\mu} x'_{\mu} = c^2 t'^2 - x'^2 - y'^2 - z'^2$$

} REMAINS INVARIANT (IS A LORENTZ INVARIANT)

$$ds^2 \equiv x'^{\mu} x'_{\mu} = x^{\mu} x_{\mu}$$

$$\Downarrow$$

$$\underline{\underline{a^{\mu}{}_{\nu}\, a_{\mu}{}^{\lambda} = g_{\nu}^{\lambda}}}$$

$$\Downarrow$$

$$\left(\det(a^{\mu}{}_{\nu})\right)^2 = 1$$

$\det(a^{\mu}{}_{\nu})$ $\nearrow$ $= +1$ PROPER LORENTZ TF (e.g. ROTATION, LORENTZ BOOST)

$\det(a^{\mu}{}_{\nu})$ $\searrow$ $= -1$ IMPROPER LORENTZ TF (e.g. INVOLVES SPACE INVERSION, OR TIME REVERSAL) e.g. ⇒ LORENTZ BOOST ALONG X-AXIS

$$\beta = \frac{v}{c}$$

$$a^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

⇒ SPACE INVERSION

$$a^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

⇒ TIME REVERSAL

$$a^\mu{}_\nu = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & +1 & \\ & & & +1 \end{pmatrix}$$

↳ PROOF OF LORENTZ COVARIANCE

SYSTEM S: $\left(i\hbar \gamma^\mu \partial_\mu - m_0 c \right) \Psi(x) = 0$

UNDER LORENTZ TF. $S \to S'$ $\quad x'^\mu = a^\mu{}_\nu \, x^\nu$

SYSTEM S': $\left(i\hbar \gamma'^\mu \frac{\partial}{\partial x'^\mu} - m_0 c \right) \Psi'(x') = 0$

DIRAC EQ: SAME FORM

* γ'^μ IN SYSTEM S' HAVE TO SATISFY

$$\gamma'^\mu \gamma'^\nu + \gamma'^\nu \gamma'^\mu = 2 g^{\mu\nu} \mathbb{1}$$

ALL 4×4 MATRICES SATISFYING ANTI-COMMUTATION RELATIONS ARE UNITARY EQUIVALENT, i.e.

$$\gamma'^\mu = U \gamma^\mu U^{-1}$$ WITH U: UNITARY TF $\quad U^{-1} = U^+$

AS UNITARY TF. DO NOT CHANGE PHYSICS WE CAN CHOOSE

$$\gamma'^\mu = \gamma^\mu$$

* $\Psi'(x')$ & $\Psi(x)$ ARE RELATED BY <u>LINEAR</u> TRANSFORMATION $S(a)$

SINCE BOTH DIRAC EQ. & LORENTZ TF. ARE <u>LINEAR</u> IN COORDINATES

$$\psi'(x') = S(a)\,\psi(x)$$

$$\begin{cases} x' = a\,x \\ x = a^{-1}\,x' \end{cases}$$ (SHORTHAND FOR $x'^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu}$

$$\psi'(x') = S(a)\,\psi(x) = S(a)\,\psi(a^{-1}x')$$

OR EQUIVALENTLY

$$\psi(x) = \underline{S^{-1}(a)}\,\psi'(x') = S^{-1}(a)\,\psi'(a\,x)$$

INVERSE TF.

$$x = (a^{-1})\,x'$$

$$\Downarrow$$

$$\psi(x) = \underline{S(a^{-1})}\,\psi'(x') = S(a^{-1})\,\psi'(a\,x)$$

$\therefore$ IDENTIFICATION $\quad S(a^{-1}) = S^{-1}(a)$

$$* \quad \left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 c \right) \psi(x) = 0$$

$$\downarrow \quad \psi(x) = S^{-1}(a)\,\psi'(x')$$

$$\left[i\hbar \gamma^\mu S^{-1}(a) \frac{\partial}{\partial x^\mu} - m_0 c \, S^{-1}(a) \right] \psi'(x') = 0$$

$\downarrow$ MULTIPLY ON LEFT BY $S(a)$

$$\left[i\hbar \left(S(a)\, \gamma^\mu \, S^{-1}(a) \right) \frac{\partial}{\partial x^\mu} - m_0 c \right] \psi'(x') = 0$$

$$\downarrow \quad \frac{\partial}{\partial x^\mu} = \underbrace{\frac{\partial x'^\nu}{\partial x^\mu}}_{a^\nu{}_\mu} \frac{\partial}{\partial x'^\nu} \qquad x'^\nu = a^\nu{}_\lambda \, x^\lambda$$

$$\left[i\hbar \underbrace{\left(S(a)\, \gamma^\mu \, S^{-1}(a) \right) a^\nu{}_\mu}_{} \; \partial'_\nu - m_0 c \right] \psi'(x') = 0$$

WE REQUIRE THIS
TO BE γ^ν
SO THAT DIRAC EQ. IN TRANSFORMED FRAME
HAS SAME FORM

$$\left(i\hbar \, \gamma^\nu \partial'_\nu - m_0 c \right) \psi'(x') = 0$$

$$\therefore \quad S(a)\,\gamma^\mu\,S^{-1}(a)\,a^\nu{}_\mu = \gamma^\nu$$

OR EQUIVALENTLY

$$\boxed{S^{-1}(a)\,\gamma^\nu\,S(a) = a^\nu{}_\mu\,\gamma^\mu}$$

↳ INFINITESIMAL PROPER LORENTZ TF

$$a^\nu{}_\mu = g^\nu{}_\mu + \Delta\omega^\nu{}_\mu$$

→ INFINITESIMAL (i.e. NEGLECT TERMS QUADRATIC IN $\Delta\omega$)

$$a^\mu{}_\nu\, a_\mu{}^\lambda = g_\nu{}^\lambda$$

$$\left(g^\mu{}_\nu + (\Delta\omega)^\mu{}_\nu\right)\left(g_\mu{}^\lambda + (\Delta\omega)_\mu{}^\lambda\right) = g^\lambda_\nu$$

$$g^\lambda_\nu + (\Delta\omega)^\lambda{}_\nu + (\Delta\omega)_\nu{}^\lambda \simeq g^\lambda_\nu$$

$$\Downarrow$$

$$(\Delta\omega)_{\mu\nu} = -(\Delta\omega)_{\nu\mu}$$

ANTI-SYMMETRIC

↓ 6 INDEPENDENT INFINITESIMAL PROPER LORENTZ TF

3 ROTATIONS

e.g. $(\Delta\omega)^1{}_2 = (\Delta\varphi) = -(\Delta\omega)^{12}$

ROTATION AROUND Z-AXIS OVER ANGLE $\Delta\varphi$

3 LORENTZ BOOSTS.

e.g. $(\Delta\omega)^0{}_1 = -(\Delta\omega)^{01} = -\Delta\beta$

BOOST ALONG X-AXIS WITH VELOCITY $c(\Delta\beta)$

$S(\Delta\omega)$ CAN BE CONSTRUCTED
BY EXPANDING UP TO TERMS <u>LINEAR</u> IN $\Delta\omega$

$$S \equiv \mathbb{1} - \frac{i}{4} \underbrace{\sigma_{\mu\nu}} (\Delta\omega)^{\mu\nu}$$

DEFINES 6 INDEPENDENT 4×4 MATRICES
WITH $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$

$$S^{-1}(a)\, \gamma^{\nu}\, S(a) = a^{\nu}{}_{\mu}\, \gamma^{\mu}$$

$\Downarrow$ FOR INFINITESIMAL TF.

$$\left[1 + \frac{i}{4} \sigma_{\alpha\beta} (\Delta\omega)^{\alpha\beta} \right] \gamma^{\nu} \left[1 - \frac{i}{4} \sigma_{\alpha\beta} (\Delta\omega)^{\alpha\beta} \right]$$

$$= \gamma^{\nu} + (\Delta\omega)^{\nu}{}_{\mu}\, \gamma^{\mu}$$

$$\Downarrow$$

$$\frac{i}{4} \left[\sigma_{\alpha\beta} , \gamma^{\nu} \right] (\Delta\omega)^{\alpha\beta} = (\Delta\omega)^{\nu}{}_{\mu}\, \gamma^{\mu}$$

$$= (\Delta\omega)^{\alpha\beta} \left[g^{\nu}_{\alpha}\, \gamma_{\beta} \right]$$

$$= \frac{1}{2} (\Delta\omega)^{\alpha\beta} \left[g^{\nu}_{\alpha}\, \gamma_{\beta} - g^{\nu}_{\beta}\, \gamma_{\alpha} \right]$$

BECAUSE $(\Delta\omega)^{\alpha\beta}$ IS ANTI-SYMM.

$\forall\, (\Delta\omega)^{\alpha\beta}$ $\Downarrow$

$$\left[\sigma_{\alpha\beta} , \gamma^{\nu} \right] = -2i \left[g^{\nu}_{\alpha}\, \gamma_{\beta} - g^{\nu}_{\beta}\, \gamma_{\alpha} \right]$$

DIRAC EQ. IS LORENTZ COVARIANT (FOR PROPER LORENTZ TF)

IF WE FIND 4×4 MATRIX $\sigma_{\alpha\beta}$ SATISFYING

$$\left[\sigma_{\alpha\beta}\,,\ \gamma^{\nu}\right] = -2i\left[g^{\nu}_{\alpha}\,\gamma_{\beta} - g^{\nu}_{\beta}\,\gamma_{\alpha}\right]$$

WITH $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$

<u>SOLUTION</u> $\quad \underline{\underline{\sigma_{\alpha\beta} = \frac{i}{2}\left[\gamma_{\alpha}, \gamma_{\beta}\right]}}$

<u>PROOF</u>

$$
\begin{aligned}
&\left[\sigma_{\alpha\beta}\,,\ \gamma^{\nu}\right] \\
&= \sigma_{\alpha\beta}\,\gamma^{\nu} - \gamma^{\nu}\,\sigma_{\alpha\beta} \\
&= \frac{i}{2}\left\{\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)\gamma^{\nu} - \gamma^{\nu}\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)\right\} \\
&= \frac{i}{2}\left\{\left(\gamma_{\alpha}\gamma_{\beta} - \gamma_{\beta}\gamma_{\alpha}\right)\gamma^{\nu} - \left(2g^{\nu}_{\alpha}\,\gamma_{\beta} - \gamma_{\alpha}\gamma^{\nu}\gamma_{\beta}\right)\right. \\
&\qquad\qquad \left. + \left(2g^{\nu}_{\beta}\,\gamma_{\alpha} - \gamma_{\beta}\gamma^{\nu}\gamma_{\alpha}\right)\right\} \\
&= \frac{i}{2}\left\{\left(\cancel{\gamma_{\alpha}\gamma_{\beta}} - \cancel{\gamma_{\beta}\gamma_{\alpha}}\right)\cancel{\gamma^{\nu}} - 2g^{\nu}_{\alpha}\,\gamma_{\beta} + 2g^{\nu}_{\beta}\,\gamma_{\alpha} - \cancel{\gamma_{\alpha}\gamma_{\beta}\gamma^{\nu}}\right. \\
&\qquad\qquad \left. + 2g^{\nu}_{\beta}\,\gamma_{\alpha} - 2g^{\nu}_{\alpha}\,\gamma_{\beta} + \cancel{\gamma_{\beta}\gamma_{\alpha}\gamma^{\nu}}\right\} \\
&= 2i\left\{g^{\nu}_{\beta}\,\gamma_{\alpha} - g^{\nu}_{\alpha}\,\gamma_{\beta}\right\} \\
&\overset{\checkmark}{\doteq} -2i\left\{g^{\nu}_{\alpha}\,\gamma_{\beta} - g^{\nu}_{\beta}\,\gamma_{\alpha}\right\}.
\end{aligned}
$$

$$\boxed{\begin{aligned} a^{\nu}{}_{\mu} &= g^{\nu}{}_{\mu} + (\Delta\omega)^{\nu}{}_{\mu} \\ S(\Delta\omega) &= \mathbb{1} + \frac{1}{8}\left[\gamma_{\mu}, \gamma_{\nu}\right](\Delta\omega)^{\mu\nu} \end{aligned}}$$

DIRAC EQ. WILL BE LORENTZ COVARIANT
IF UNDER LORENTZ TF $x' = a\,x$
THE SPINORS TRANSFORM AS $\psi'(x') = S(a)\,\psi(x)$
WITH S AS SHOWN ABOVE

↳ FINITE PROPER LORENTZ TRANSFORMATION

* WRITE INFINITESIMAL TF AS

$$(\Delta\omega)^{\mu}{}_{\nu} = (\Delta\omega)\,(I_n)^{\mu}{}_{\nu}$$

WITH $(\Delta\omega)$ INFINITESIMAL 'ANGLE' ABOUT AXIS n

e.g. FOR BOOST ALONG x- AXIS $\Delta\omega = (\Delta\beta\ )$

$$(I_1)^{\mu}{}_{\nu} = \left(\begin{array}{cc|c} 0 & -1 & \\ -1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right)$$

$$(I_1)^{0}{}_{1} = (I_1)^{1}{}_{0} = -1$$

$$(I_1)^2 = \left(\begin{array}{cc|c} 1 & & \\ & 1 & 0 \\ \hline & 0 & 0 \end{array}\right) \qquad \begin{aligned} (I_1)^3 &= I_1 \\ (I_1)^4 &= I_1^2 \\ &\ \vdots \end{aligned}$$

* FINITE PROPER LORENTZ TF

CAN BE WRITTEN AS SUCCESSION OF INFINITESIMAL ONES

$$x'^{\mu} = \lim_{N\to\infty} \left(\mathbb{1} + \frac{\omega}{N} I_n\right)^{\mu}{}_{\alpha_1} \cdot \left(\mathbb{1} + \frac{\omega}{N} I_n\right)^{\alpha_1}{}_{\alpha_2} \cdot \ldots$$

$$\ldots \left(\mathbb{1} + \frac{\omega}{N} I_n\right)^{\alpha_{N-1}}{}_{\alpha_N} \; x^{\alpha_N}$$

NOTE $\Delta\omega = \frac{\omega}{N}$ WITH N LARGE

$$= \lim_{N\to\infty} \left[\left(\mathbb{1} + \frac{\omega}{N} I_n\right)^N\right]^{\mu}{}_{\nu} \cdot x^{\nu}$$

$$= \left(e^{\omega I_n}\right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

$$= \Bigg(\underbrace{\left[1 + \frac{(\omega I_n)^2}{2!} + \frac{(\omega I_n)^4}{4!} + \ldots\right]}_{\text{EVEN POWERS}} + \underbrace{\left[\omega I_n + \frac{(\omega I_n)^3}{3!} + \ldots\right]}_{\text{ODD POWERS}} \Bigg)^{\mu}{}_{\nu} \cdot x^{\nu}$$

$$= \left(\cosh(\omega I_n) + \sinh(\omega I_n)\right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

$\downarrow$ BECAUSE $I_n^4 = I_n^2$

$I_n^3 = I_n$

$$= \left(\mathbb{1} - I^2 + (\cosh\omega)\, I^2 + (\sinh\omega)\, I\right)^{\mu}{}_{\nu} \cdot x^{\nu}$$

e.g. BOOST ALONG X- AXIS

$$\mathbb{1} - (I_1)^2 + (\cosh\omega)(I_1)^2 + (\sinh\omega)\, I_1$$

$$= \left[\begin{array}{cc|cc} \cosh\omega & -\sinh\omega & & \\ -\sinh\omega & \cosh\omega & \multicolumn{2}{c}{0} \\ \hline & & 1 & \\ \multicolumn{2}{c|}{0} & & 1 \end{array}\right]$$

IDENTIFY

$$\sinh\omega = \beta\gamma$$

$$\cosh\omega = \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

* FINITE SPINOR TRANSFORMATION

$$\psi'(x') = S(a)\,\psi(x)$$

$$= \lim_{N\to\infty}\left(1 - \frac{i\omega}{4N}\sigma_{\alpha\beta}(I_n)^{\alpha\beta}\right)^N \psi(x)$$

$$\boxed{\psi'(x') = \exp\left\{-\frac{i}{4}\,\omega\,\sigma_{\alpha\beta}(I_n)^{\alpha\beta}\right\}\psi(x)}$$

$\rightsquigarrow$ e.g. LORENTZ BOOST ALONG X-AXIS.

$$(\underline{I}_1)^{01} = 1$$
$$(\underline{I}_1)^{10} = -1$$

$$\psi'(x') = \exp\left\{-\frac{i}{2}\,\omega\;\sigma_{01}\right\}\psi(x)$$

WITH
$$\sigma_{01} = \frac{i}{2}\left(\gamma_0\gamma_1 - \gamma_1\gamma_0\right)$$
$$= -i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0\end{pmatrix} = -i\,\alpha_1$$

$\rightsquigarrow$ ROTATION AROUND Z-AXIS. AROUND ANGLE φ

$$(\underline{I})^1{}_2 = 1 \qquad (\underline{I})^2{}_1 = -1$$

$$(\underline{I})^{12} = -1 \qquad (\underline{I})^{21} = +1$$

$$\psi'(x') = \exp\left\{+\frac{i}{2}\varphi\,\sigma_{12}\right\}\psi(x)$$

WITH
$$\sigma_{12} = \frac{i}{2}\left(\gamma_1\gamma_2 - \gamma_2\gamma_1\right) = \begin{pmatrix}\sigma_3 & 0 \\ 0 & \sigma_3\end{pmatrix}$$
$$= \Sigma_3$$

ANALOGY WITH 2-COMPONENT SPINORS φ (PAULI)
$$\varphi'(x') = \exp\left\{\frac{i}{2}\,\varphi\;\sigma_3\right\}\varphi(x)$$

NOTE CONSIDER SPINOR WITH SPIN PROJ $+\frac{\hbar}{2}$ ALONG Z- AXIS

i.e. $\Sigma_3 \psi = + \psi$ EIGENSTATE OF Σ_3

UNDER ROTATION, ψ TRANSFORMS AS

$$\psi'(x') = \exp\left(\frac{i}{2}\varphi\right) \psi(x)$$

$\therefore$ IT TAKES A ROTATION OVER $\underline{\underline{4\pi}}$ (!) BEFORE SPINOR TURNS INTO ITSELF

(DOUBLE VALUEDNESS OF SPINOR LAW OF ROTATION)

$\Downarrow$

PHYSICAL QUANTITIES MUST BE <u>BILINEARS</u> IN ψ

↳ TURN INTO THEMSELVES AFTER ROTATION OVER 2π

NOTE : FOR ROTATIONS $\sigma_{ij}^{+} = \sigma_{ij}$

$$\Rightarrow \quad S_R^{+} = S_R^{-1} \quad \text{UNITARY}$$

FOR LORENTZ BOOSTS.

$$S_B^{+} = \exp\left\{-\frac{\omega}{2}\alpha_1\right\} = S_B$$

$$S_B^{-1} = \gamma_0 S_B^{+} \gamma_0$$

(SHOW THIS YOURSELF !)

$\therefore$ FOR PROPER LORENTZ TF $\underline{\underline{S^{-1} = \gamma_0 S^{+} \gamma_0}}$

COVARIANCE OF CONTINUITY EQUATION

↳ CONTINUITY EQUATION

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} . \bar{J} = 0$$

$$\rho = \Psi^{+} \psi \quad , \quad \bar{J} = c\, \psi^{+} \bar{\alpha} \psi$$

$$\rho = \psi^{+} \gamma^{0} \gamma^{0} \psi \quad , \quad J^{i} = c\, \psi^{+} \gamma^{0} \gamma^{i} \psi$$

NOTATION "ψ BAR" $\bar{\psi} \equiv \psi^{+} \gamma^{0}$

$$J^{\mu} \equiv (c\rho , \bar{J})$$

$$\boxed{J^{\mu}(x) = c \bar{\psi}(x)\, \gamma^{\mu} \psi(x)}$$

CONT. EQUATION : $\frac{\partial}{\partial x^{\mu}} J^{\mu} = 0$

↳ COVARIANCE

$$J'^{\mu}(x') = c \bar{\psi}'(x')\, \gamma^{\mu} \psi'(x')$$

$$= c\, \psi^{+}(x)\, S^{+} \gamma^{0} \gamma^{\mu} S\, \psi(x)$$

$\downarrow$ USING $S^{-1} = \gamma_{0} S^{+} \gamma_{0}$

$$= c\, \underbrace{\psi^{+}(x)\, \gamma_{0}}_{\bar{\psi}} \underbrace{S^{-1} \gamma^{\mu} S}_{a^{\mu}{}_{\nu} \gamma^{\nu}}\, \psi(x).$$

$$= a^{\mu}{}_{\nu}\, c \bar{\psi}(x)\, \gamma^{\nu} \psi(x) = a^{\mu}{}_{\nu}\, J^{\nu}(x)$$

$\therefore$ J^{μ} TRANSFORMS AS A FOUR-VECTOR.

- SPATIAL INVERSION

↳ IMPROPER LORENTZ TF

$$\bar{x}' = -\bar{x}$$
$$t' = t$$

$$a^{\mu}{}_{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

↳ DENOTE SPINOR TF S BY P:

i.e. $\psi'(x') = P\,\psi(x)$

$$P^{-1}\gamma^{\nu}P = a^{\nu}{}_{\mu}\gamma^{\mu}.$$

$\nu = 0$ $\quad P^{-1}\gamma^{0}P = \gamma^{0}$

$\nu = i$ $\quad P^{-1}\gamma^{i}P = -\gamma^{i}$

SATISFIED IF $\underline{\underline{P = \gamma^{0}\underbrace{e^{i\varphi}}}}$

PHASE FACTOR OF NO PHYSICAL IMPORTANCE

$$\therefore \quad \psi'(t,-\bar{x}) = e^{i\varphi}\gamma^{0}\psi(t,\bar{x})$$

• BILINEAR COVARIANTS

↳ PHYSICAL QUANTITIES CORRESPOND WITH BILINEAR EXPRESSIONS OF SPINORS WITH 4 x 4 MATRIX INBETWEEN

i.e. $\bar{\psi}\,\Gamma\,\psi$

THERE ARE 16 INDEPENDENT BILINEAR COVARIANTS

SCALAR (1) $\Gamma_S = \mathbb{1}$

VECTOR (4) $\Gamma_V^\mu = \gamma^\mu$

AXIAL VECTOR (4) $\Gamma_A^\mu = \gamma^\mu \gamma_5$

PSEUDO-SCALAR (1) $\Gamma_P = i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma_5 = \gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

TENSOR (6) $\Gamma_T^{\mu\nu} = \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$

↳ PROPERTIES

i) $\forall \Gamma$: $\Gamma^2 = \pm \mathbb{1}$

$\rightsquigarrow \Gamma_S^2 = \mathbb{1}$

$\rightsquigarrow \Gamma_P^2 = \mathbb{1}$ $\quad (\gamma_5)^2 = \mathbb{1}$

$$\left.\begin{aligned} \rightsquigarrow (\gamma^0)^2 &= \mathbb{1} \\ (\gamma^i)^2 &= -\mathbb{1} \end{aligned}\right\} \quad (\gamma^\mu)^2 = \mathbb{1}\, g^{\mu\mu}$$

(NOT SUMMED OVER μ HERE!)

$\rightsquigarrow \gamma^\mu \gamma_5 + \gamma_5 \gamma^\mu = 0$

$$(\gamma^\mu \gamma_5)^2 = \gamma^\mu \gamma_5 \, \gamma^\mu \gamma_5 \qquad \text{(NOT SUMMED OVER)}$$

$$= - \gamma^\mu \gamma_5 \gamma_5 \gamma^\mu \qquad \gamma_5^2 = \mathbb{1}$$

$$= -(\gamma^\mu)^2$$

$$= - g^{\mu\mu} \, \mathbb{1}$$

$$\rightsquigarrow \quad (\Gamma_T^{\mu\nu})^2 = -\frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$

$$= -\frac{1}{4}\Big\{ \gamma^\mu\gamma^\nu\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\nu\gamma^\mu + \gamma^\nu\gamma^\mu\gamma^\nu\gamma^\mu \Big\}$$

$$= -\frac{1}{4}\Big\{ 2g^{\nu\mu}\gamma^\mu\gamma^\nu - (\gamma^\mu)^2(\gamma^\nu)^2 - g^{\mu\mu}(\gamma^\nu)^2 - g^{\nu\nu}(\gamma^\mu)^2 + 2g^{\nu\mu}\gamma^\nu\gamma^\mu - (\gamma^\nu)^2(\gamma^\mu)^2 \Big\}$$

$$= -\frac{1}{4}\Big\{ 4g^{\mu\nu} g^{\mu\nu} - 4 g^{\mu\mu} g^{\nu\nu} \Big\}$$

$$= \left[g^{\mu\mu} g^{\nu\nu} - (g^{\mu\nu})^2 \right] \mathbb{1}$$

$$= \begin{cases} 0 & \mu = \nu \\ g^{\mu\mu} g^{\nu\nu} \mathbb{1} & \mu \neq \nu \end{cases}$$

ii) $\forall \Gamma_i$ EXCEPT Γ_S :

$\exists\, \Gamma_j$ SUCH THAT $\Gamma_i \Gamma_j = - \Gamma_j \Gamma_i$

$$\Downarrow$$

$$\underline{\underline{\mathrm{Tr}\, \Gamma_i = 0}}$$

PROOF : $\rightsquigarrow \mathrm{Tr}\, \gamma_5 = 0$

$\rightsquigarrow \mathrm{Tr}\, \gamma^\mu = 0$

$\rightsquigarrow \mathrm{Tr}\, \gamma^\mu \gamma_5 = 0$

$$\gamma^i \gamma_5 = \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$\gamma^0 \gamma_5 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

$\rightsquigarrow \mathrm{Tr}\, \sigma^{\mu\nu} = 0$

$$\mathrm{Tr}\left\{ \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right\}$$

$$= \underbrace{\mathrm{Tr}\left\{ \gamma^\mu \gamma^\nu \right\}}_{= \mathrm{Tr}\left\{ \gamma^\nu \gamma^\mu \right\}} - \mathrm{Tr}\left\{ \gamma^\nu \gamma^\mu \right\}$$

$$= 0$$

iii) $\forall\, \Gamma_i, \Gamma_j \quad (i \neq j)$

$\exists\, \Gamma_k \; (\neq \Gamma_S)$ SUCH THAT

$$\Gamma_i \Gamma_j = c\, \Gamma_k \qquad c : \text{COMPLEX NUMBER}$$

CHECK THIS BY DIRECT INSPECTION

iv) Γ_i ARE LINEARLY INDEPENDENT

i.e. IF $\sum_i a_i \Gamma_i = 0$

$$\Updownarrow$$

$$a_i = 0$$

$\rightsquigarrow$ $\sum_i a_i \Gamma_i = 0$

$\downarrow$ MULTIPLY BY Γ_j $(j \neq S)$

$$\sum_i a_i \Gamma_j \Gamma_i = 0$$

$\downarrow$ TAKE Tr

$$0 = \sum_i a_i \operatorname{Tr}\{\Gamma_j \Gamma_i\} = a_j \operatorname{Tr}\{\Gamma_j^2\} + \sum_{i \neq j} a_i \underbrace{\operatorname{Tr}\{\Gamma_j \Gamma_i\}}_{= 0}$$

$$= \pm 4 a_j$$

$$\Updownarrow$$

$$\underline{a_j = 0}$$

$\rightsquigarrow$ FOR $\Gamma_j = \Gamma_S$

$$\sum_i a_i \Gamma_S \Gamma_i = 0$$

$$\Downarrow$$

$$\sum_i a_i \operatorname{Tr}\{\Gamma_S \Gamma_i\} = 0$$

$$a_S \underbrace{\operatorname{Tr}\{\Gamma_S^2\}}_{4} + \sum_{i \neq S} a_i \underbrace{\operatorname{Tr}\{\Gamma_i\}}_{0} = 0$$

$$\Updownarrow$$

$$\underline{a_S = 0}$$

LORENTZ TRANSFORMATION OF BILINEAR COVARIANTS

- S (SCALAR)

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x)$$

PROOF
$$\psi'(x') = S\psi(x)$$
$$\bar{\psi}'(x') = \bar{\psi}(x)S^{-1}$$

- P (PSEUDO SCALAR)

$$\bar{\psi}'(x')\gamma_5\psi'(x') = (\det a)\,\bar{\psi}(x)\gamma_5\psi(x)$$

PROOF
$$\bar{\psi}'(x')\gamma_5\psi'(x')$$
$$= \bar{\psi}(x)S^{-1}\gamma_5 S\psi(x)$$

PROPER: $S = \exp\left\{-\frac{i}{4}\omega\,\sigma_{\alpha\beta}\,(I_n)^{\alpha\beta}\right\}$

$$[\gamma_5, \sigma_{\alpha\beta}] = 0$$
$$\Downarrow$$
$$\gamma_5 S = S\gamma_5$$
$$S^{-1}\gamma_5 S = \gamma_5$$

IMPROPER $S = P = \gamma^0$
$$S^{-1}\gamma_5 S = \gamma^0\gamma_5\gamma^0 = -\gamma_5(\gamma^0)^2 = -\gamma_5$$

- **V (VECTOR)**

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = a^\mu{}_\nu\,\bar{\psi}(x)\gamma^\nu\psi(x)$$

PROOF $\bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}(x)\underbrace{S^{-1}\gamma^\mu S}_{a^\mu{}_\nu\gamma^\nu}\psi(x)$

$$= a^\mu{}_\nu\,\bar{\psi}(x)\gamma^\nu\psi(x)$$

- **A (AXIAL - VECTOR)**

$$\bar{\psi}'(x')\gamma^\mu\gamma_5\psi'(x') = (\det a)\,a^\mu{}_\nu\,\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)$$

PROOF $\bar{\psi}'(x')\gamma^\mu\gamma_5\psi'(x') = \bar{\psi}(x)S^{-1}\gamma^\mu\underbrace{\gamma_5 S}_{S\gamma_5(\det a)}\psi(x)$

$$= a^\mu{}_\nu(\det a)\,\bar{\psi}(x)\gamma^\nu\gamma_5\psi(x)$$

- **T (TENSOR)**

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = a^\mu{}_\kappa\,a^\nu{}_\lambda\,\bar{\psi}(x)\sigma^{\kappa\lambda}\psi(x)$$

PROOF $\bar{\psi}'(x')\frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi'(x')$

$$= \bar{\psi}(x)\frac{i}{2}S^{-1}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)S\psi(x)$$

$$= \bar{\psi}(x)\frac{i}{2}\Big(\underbrace{S^{-1}\gamma^\mu S}_{a^\mu{}_\kappa\gamma^\kappa}\underbrace{S^{-1}\gamma^\nu S}_{a^\nu{}_\lambda\gamma^\lambda} - S^{-1}\gamma^\nu S\,S^{-1}\gamma^\mu S\Big)\psi(x)$$

$$= a^\mu{}_\kappa\,a^\nu{}_\lambda\,\bar{\psi}(x)\frac{i}{2}(\gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^\kappa)\psi(x)$$

- PLANE-WAVE SOLUTIONS OF DIRAC EQUATION CONSTRUCTED FROM LORENTZ TRANSFORMATION

↳ FREE DIRAC PARTICLE AT REST

$$p^{\mu}_{(0)}\left(\frac{E}{c}, \bar{p}\right) = (m_0 c, 0)$$

↑ IN REST FRAME

$$\psi_r(x) = w_r(0)\, e^{-\frac{i}{\hbar}\lambda_r (m_0 c^2) t}$$

$$\lambda_r = \begin{cases} +1 & , r = 1,2 \\ -1 & , r = 3,4 \end{cases}$$

$$w_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad w_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$w_3(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad w_4(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{cases} \Sigma_3 w_1 = + w_1 \\ \Sigma_3 w_2 = - w_2 \\ \Sigma_3 w_3 = + w_3 \\ \Sigma_3 w_4 = - w_4 \end{cases}$$

w_r : EIGENFUNCTION OF Z-COMP. OF "SPIN OPERATOR"

$$\bar{\Sigma} = \begin{pmatrix} \bar{\sigma} & 0 \\ 0 & \bar{\sigma} \end{pmatrix}$$

↳ TO DESCRIBE FREE PARTICLE WITH <u>FINITE</u> MOMENTUM $\bar{p}$

$$p^\mu \quad \left(\frac{E}{c}, \bar{p}\right) \qquad\qquad E^2 = c^2\bar{p}^2 + m_0^2 c^4$$

PERFORM LORENTZ BOOST ON DIRAC SPINOR AT REST

$$w_r(\bar{p}) = S \; w_r(0)$$

$$\rightsquigarrow \quad S = \exp\left\{-\frac{i}{2}\omega\,\sigma_{01}\right\}$$ (FOR LORENTZ BOOST ALONG x-AXIS)

$$\downarrow \quad \sigma_{01} = -i\alpha_1$$

$$= \exp\left\{-\frac{\omega}{2}\alpha_1\right\}$$

$$= \cosh\frac{\omega}{2}\,\mathbb{1}_{4\times4} - \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\sinh\frac{\omega}{2}$$

$$= \cosh\frac{\omega}{2}\left[\begin{array}{cc|cc} 1 & 0 & 0 & -\tanh\frac{\omega}{2} \\ 0 & 1 & -\tanh\frac{\omega}{2} & 0 \\ \hline 0 & -\tanh\frac{\omega}{2} & 1 & 0 \\ -\tanh\frac{\omega}{2} & 0 & 0 & 1 \end{array}\right]$$

IN SYSTEM S : PARTICLE AT REST $\bar{p} = 0$

IN SYSTEM S' (MOVING WITH SPEED $-v$ ALONG X-AXIS)

↳ PARTICLE HAS MOMENTUM $\bar{p}$

$$\left.\right\} \; -\frac{v}{c} = \tanh \omega$$

USE $$\tanh \omega = \frac{2 \tanh \omega/2}{1 + \tanh^2 \omega/2}$$

$$1 + \sqrt{1 - \tanh^2 \omega} = \frac{2}{1 + \tanh^2 \omega/2}$$

$$\Downarrow$$

$$\tanh \frac{\omega}{2} = \frac{\tanh \omega}{1 + \sqrt{1 - \tanh^2 \omega}}$$

$$\hookrightarrow \quad -\tanh \frac{\omega}{2} = \frac{v/c}{1 + \sqrt{1 - \beta^2}} = \frac{\gamma v/c}{1 + \gamma}$$

$$= \frac{\gamma m_0 v c}{m_0 c^2 + \gamma m_0 c^2}$$

$$\left.\right\} \quad \begin{aligned} p &= \gamma m_0 v \\ E &= \gamma m_0 c^2 \end{aligned}$$

$$-\tanh \frac{\omega}{2} = \frac{p c}{m_0 c^2 + E}$$

$$\cosh \frac{\omega}{2} = \frac{1}{\sqrt{1 - \tanh^2 \frac{\omega}{2}}} = \sqrt{\frac{E + m_0 c^2}{2 m_0 c^2}}$$

$$S = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \left[\begin{array}{cc|cc} 1 & & 0 & \frac{cp}{E+m_0c^2} \\ & 1 & \frac{cp}{E+m_0c^2} & 0 \\ \hline 0 & \frac{cp}{E+m_0c^2} & 1 & \\ \frac{cp}{E+m_0c^2} & 0 & & 1 \end{array}\right]$$

$$= \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \left[\begin{array}{c|c} \mathbb{1} & \frac{c}{E+m_0c^2}\sigma_x p \\ \hline \frac{c}{E+m_0c^2}\sigma_x p & \mathbb{1} \end{array}\right]$$

$\rightsquigarrow$ FOR ARBITRARY LORENTZ BOOST ALONG DIRECTION $\bar{p}$

$$S = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \left[\begin{array}{c|c} \mathbb{1} & \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p} \\ \hline \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p} & \mathbb{1} \end{array}\right]$$

$$w_r(\bar{p}) = S\, w_r(0) \Rightarrow w_1(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} 1 \\ 0 \\ \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p}\begin{pmatrix}1\\0\end{pmatrix} \end{bmatrix}$$

$$w_2(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} 0 \\ 1 \\ \frac{c}{E+m_0c^2}\bar{\sigma}\cdot\bar{p}\begin{pmatrix}0\\1\end{pmatrix} \end{bmatrix}$$

$$w_3(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} c\,\frac{\bar{\sigma}\cdot\bar{p}}{E+m_0c^2}\begin{pmatrix}1\\0\end{pmatrix} \\ 1 \\ 0 \end{bmatrix}$$

$$w_4(\bar{p}) = \sqrt{\frac{E+m_0c^2}{2m_0c^2}} \begin{bmatrix} c\,\frac{\bar{\sigma}\cdot\bar{p}}{E+m_0c^2}\begin{pmatrix}0\\1\end{pmatrix} \\ 0 \\ 1 \end{bmatrix}$$

$\rightsquigarrow$ FREE PARTICLE SOLUTION

$$(m_0c^2)t = p^{\mu}_{(0)}\, x_{\mu}^{(0)}$$

(0) DENOTES REST FRAME

$$= p^{\mu}\, x_{\mu}$$ IN ARBITRARY FRAME (LORENTZ INVARIANT!)

$$\therefore \boxed{\psi_r(x) = w_r(\bar{p})\, e^{-\frac{i}{\hbar}\lambda_r\, p_{\mu} x^{\mu}}}$$

⇝ DIRAC EQ.

$$\left(i\hbar \gamma^\mu \partial_\mu - m_0 c \right) \psi_r(x) = 0$$

$$\Downarrow$$

$$\left(\gamma^\mu p_\mu - \lambda_r m_0 c \right) w_r(\bar{p}) = 0$$

$$\Updownarrow$$

$$\underline{\underline{\left(\not{p} - \lambda_r m_0 c \right) w_r(\bar{p}) = 0}}$$

⇝ NORMALIZATION (BY DIRECT INSPECTION) CHECK !

$$\bar{w}_r(\bar{p})\, w_{r'}(\bar{p})$$

$$= w_r^+(\bar{p})\, \gamma^0\, w_{r'}(\bar{p}) = \delta_{rr'}\, \lambda_r$$

⇝ COMPLETENESS (CHECK !)

$$\sum_{r=1}^{4} \lambda_r \left(w_r(\bar{p}) \right)_\alpha \left(\bar{w}_r(\bar{p}) \right)_\beta = \delta_{\alpha\beta}$$

$\alpha, \beta = 1 \ldots 4$

↳ POLARIZATION FOR DIRAC PARTICLE

⤳ REST FRAME

ASSUME PARTICLE SPIN ALONG Z-AXIS

$$w_1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ SPIN PROJ $+\frac{\hbar}{2}$ ALONG Z-AXIS

$$w_2(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$ SPIN PROJ $-\frac{\hbar}{2}$

IN REST FRAME $p^\mu_{(0)} = (m_0 c, 0)$

INTRODUCE POLARIZATION VECTOR $s^\mu_{(0)} = (0, \bar{s})$

$\bar{s}^2 = 1$

$$\bar{\Sigma}.\bar{s}\, w_1(0) = + w_1(0)$$
$$\bar{\Sigma}.\bar{s}\, w_2(0) = - w_2(0)$$

$\bar{s}$ IS UNIT VECTOR GIVEN BY SPIN AXIS (e.g. IN ABOVE EXAMPLE $\bar{s} = \hat{e}_z$)

WE OBSERVE $s^\mu_{(0)} s_{(0)\mu} = -\bar{s}^2 = -1.$

$$s^\mu_{(0)} p_{(0)\mu} = 0$$

$\rightsquigarrow$ IN ARBITRARY FRAME

$$s^\mu = a^\mu{}_\nu \, (s_{(0)})^\nu$$

$\uparrow$ REST FRAME

$$\left.\begin{aligned} s^\mu s_\mu &= -1 \\ s^\mu p_\mu &= 0 \end{aligned}\right\}$$ BECAUSE l.h.s IS LORENTZ INV.

DEFINE SPINOR POLARIZATION IN REST FRAME

e.g. CHOOSE Z-AXIS AS SPIN AXIS

$$s = u_z$$

$$s_{(0)} = u_z^{(0)} = (0, \hat{e}_z).$$

NOTATION

	SPINOR WITH SPIN PROJ ALONG Z-AXIS IN REST FR. (Σ_3)
$u(p, u_z) \equiv w_1(\bar{p})$	$+1$
$u(p, -u_z) \equiv w_2(\bar{p})$	-1
$v(p, -u_z) \equiv w_3(\bar{p})$	$+1$
$v(p, u_z) \equiv w_4(\bar{p})$	-1

$$(\not{p} - m_0 c)\, u(p, \pm u_z) = 0$$ POS. ENERGY SOLUTIONS

$$(\not{p} + m_0 c)\, v(p, \pm u_z) = 0$$ NEG. ENERGY SOLUTIONS.

NOTE FOR NEGATIVE ENERGY SOLUTION

WE WILL INTERPRET ABSENCE OF DIRAC PARTICLE WITH $-p$ AND NEGATIVE SPIN PROJ.

AS AN ANTI-PARTICLE WITH $+p$ AND POSITIVE SPIN PROJ

$\Downarrow$

NOTATION $v(p, u_z)$

$\uparrow$ DENOTES SPIN PROJ OF ANTI-PARTICLE

PROJECTION OPERATORS. FOR ENERGY & SPIN

DEFINITIONS

$P_r(\bar{p})$ PROJECTS OUT SPINOR r

$$P_r(\bar{p})\, w_{r'}(\bar{p}) = \delta_{rr'}\, w_r(\bar{p})$$

$$P_r(\bar{p})\, P_{r'}(\bar{p}) = \delta_{rr'}\, P_r(\bar{p})$$

ENERGY PROJECTORS

$$\boxed{\Lambda_r(\bar{p}) \equiv \frac{\lambda_r \not{p} + m_0 c}{2 m_0 c}}$$

CHECK

$$\left.\begin{aligned} * \quad \Lambda_{1,2}(\bar{p}) &= \frac{\not{p} + m_0 c}{2 m_0 c} \\ * \quad \Lambda_{3,4}(\bar{p}) &= \frac{-\not{p} + m_0 c}{2 m_0 c} \end{aligned}\right\} \quad \Lambda_{\pm}(\bar{p}) \equiv \frac{\pm \not{p} + m_0 c}{2 m_0 c}$$

$$* \quad \Lambda_r(\bar{p})\, w_r(\bar{p}) = \frac{\lambda_r \not{p} + m_0 c}{2 m_0 c}\, w_r(\bar{p})$$

$$\downarrow \quad \not{p}\, w_r(\bar{p}) = \lambda_r\, m_0 c\, w_r(\bar{p})$$

$$= w_r(\bar{p})$$

$$* \quad \Lambda_r(\bar{p})\, \Lambda_{r'}(\bar{p}) = \frac{(\lambda_r \not{p} + m_0 c)(\lambda_{r'} \not{p} + m_0 c)}{4 m_0^2 c^2}$$

$$= \frac{1}{4 m_0^2 c^2} \left\{ \lambda_r \lambda_{r'} \not{p}\not{p} + (\lambda_r + \lambda_{r'})\, m_0 c \not{p} + m_0^2 c^2 \right\}$$

$$\downarrow \quad \not{p}\not{p} = \gamma_\mu \gamma_\nu p^\mu p^\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu)\, p^\mu p^\nu$$

$$= g_{\mu\nu}\, p^\mu p^\nu = p_\mu p^\mu$$

$$= \frac{E^2}{c^2} - \bar{p}^2 = m_0^2 c^2$$

$$= \frac{1}{4 m_0^2 c^2} \left\{ (\lambda_r \lambda_{r'} + 1)\, m_0^2 c^2 + (\lambda_r + \lambda_{r'})\, m_0 c \not{p} \right\}$$

$$= \frac{(1 + \lambda_r \lambda_{r'})}{2} \; \frac{(\lambda_r \not{p} + m_0 c)}{2 m_0 c}.$$

$$= \left(\frac{1 + \lambda_r \lambda_{r'}}{2} \right) \Lambda_r(\bar{p})$$

$$\therefore \quad \boxed{\begin{aligned} \Lambda_\pm^2(p) &= \Lambda_\pm(p) \\ \Lambda_+ \Lambda_- &= 0 \\ \Lambda_+ + \Lambda_- &= \mathbb{1} \end{aligned}}$$

↳ SPIN PROJECTORS (FOR SPIN VECTOR s^μ)

$$\boxed{\Sigma(s) \equiv \frac{1 + \gamma_5 \not{s}}{2}}$$

IN REST FRAME $\not{s} = \gamma_\mu s^\mu$ (LORENTZ INV.)

$$\longrightarrow = - \bar{\gamma} . \bar{s}$$

* IF SPIN AXIS // Z-AXIS : $\bar{s} = \hat{e}_z$

$$\Sigma(u_z) = \frac{1 + \gamma_5(-\gamma^3)}{2}$$

$$= \frac{1 + \Sigma^3 \gamma_0}{2} = \frac{1}{2}\left\{ \mathbb{1}_{4\times4} + \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right\}$$

$$\Sigma(u_z)\, u(p, u_z) = u(p, u_z)$$

$$\Sigma(u_z)\, v(p, u_z) = v(p, u_z)$$

BECAUSE $\frac{1 + \Sigma^3 \gamma_0}{2} w_{1,2}(0) = \frac{1 + \Sigma^3}{2} w_{1,2}(0)$

$$= \begin{cases} 1 & , w_1 \\ 0 & , w_2 \end{cases}$$

$$\frac{1 + \Sigma^3 \gamma_0}{2} w_{3,4}(0) = \frac{1 - \Sigma^3}{2} w_{3,4}(0)$$

$$= \begin{cases} 0 & , w_3 \\ 1 & , w_4 \end{cases}$$

$$\Sigma(-u_z)\; u(p, u_z) = 0$$

$$\Sigma(-u_z)\; v(p, u_z) = 0$$

BECAUSE $\quad \dfrac{1 - \Sigma^3 \gamma_0}{2}\, w_{3,4}(0) = \dfrac{1 + \Sigma^3}{2}\, w_{3,4}(0)$

$$= \begin{cases} 1 , & w_3 \\ 0 , & w_4 \end{cases}$$

* FOR ARBITRARY SPIN AXIS

$$\Sigma(s) = \frac{1 + \gamma_5 \not{s}}{2}$$

$$\Sigma(s)\; u(p,s) = u(p,s)$$

$$\Sigma(s)\; v(p,s) = v(p,s)$$

$$\Sigma(-s)\; u(p,s) = \Sigma(-s)\; v(p,s) = 0 .$$

↳ SIMULTANEOUS ENERGY & SPIN PROJECTORS

$$P_1(\bar{p}) = \Lambda_+(\bar{p})\; \Sigma(u_z)$$

$$P_2(\bar{p}) = \Lambda_+(\bar{p})\; \Sigma(-u_z)$$

$$P_3(\bar{p}) = \Lambda_-(\bar{p})\; \Sigma(-u_z)$$

$$P_4(\bar{p}) = \Lambda_-(\bar{p})\; \Sigma(+u_z)$$