4 problems. Total number of points given is 26.
Grading: 20+ excellent, 15+ good, below 10 is incomplete.
Deadline: Thurdsay, November 21.

1. **Compton scattering in scalar QED**

   Consider the scalar QED theory with following Lagrangian density:

   $$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\Box + m^2)\phi - ieA^\mu(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*) + e^2 A_\mu^2 |\phi|^2.$$

   The complex scalar field $\phi(x)$ describes a charged particle with spin 0, let's say, the pion $\pi^\pm$.

   Consider the Compton scattering (i.e., $\gamma\pi \to \gamma\pi$). Let the initial momenta of the pion and photon be p and q respectively, and the final ones p' and q'.

   (a) {3 pts} Calculate the amplitude squared, $|\mathcal{M}|^2$, for the unpolarized Compton scattering at tree level (see Fig.1).

   Hint: Use the completeness relation for the photon polarization vectors:

   $$\sum_{\text{pols}} \varepsilon^\mu(q)\,\varepsilon^{*\,\nu}(q) = -g^{\mu\nu}.$$

   Express the result in terms of these two kinematical invariants:

   $$\nu = \frac{p \cdot q}{m} = \frac{s - m^2}{2m}, \qquad \nu' = \frac{p \cdot q'}{m} = \frac{m^2 - u}{2m},$$

   which in the lab frame (i.e., pion rest-frame) have the interpretation of the photon energy before and after scattering.
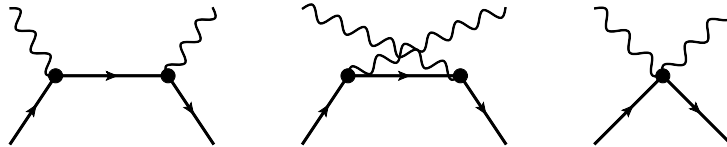

Figure 1: Diagrams of scalar QED that contribute to the tree-level Compton scattering process.

(b) {3 pts} Check your result using FORM (provide the print out of your FORM code).

(c) {2 pts} Introduce the scattering angle in the lab frame, θ. Show that the unpolarized angular distribution (i.e., the differential cross section) in the lab frame is written as:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2} \frac{1 - \frac{1}{2}\sin^2\theta}{\left[1 + \frac{\nu}{m}(1 - \cos\theta)\right]^2},$$

where $\alpha = e^2/4\pi$. *Hint:* use the following kinematical relations: $t = -2m(\nu - \nu') = -2\nu\nu'(1 - \cos\theta)$, $dt = (\nu'^2/\pi)d\Omega$.

(d) {2 pts} Calculate the total unpolarized cross section of this process, $\sigma(\nu)$.
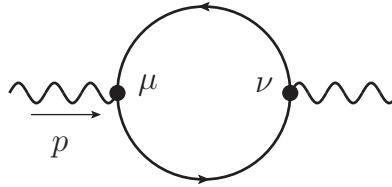
2. **Vacuum polarization in QED**

Figure 2: Vacuum polarization diagram in QED

Consider the vacuum polarization tensor $\Pi^{\mu\nu}(p)$ (see Fig.2).

(a) {1 pt}Using Lorenz symmetry and gauge invariance, show that it can be written as:

$$\Pi^{\mu\nu}(p^2) = [g^{\mu\nu}p^2 - p^\mu p^\nu]\Pi(p^2), \qquad (1)$$

where $\Pi(p^2)$ is a scalar function.

(b) {2 pts} Find the expression for $\Pi(p^2)$ in terms of the Passarino-Veltman integrals (used in LoopTools).

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(c) {1 pt} Calculate the result for $p^2 = 6m^2$ in Mathematica. Why is the result complex? and what is the threshold for the appearance of the imaginary part?

(d) {2 pts} What is the expression for the renormalized photon propagator, dressed by the vacuum polarization.

3. **Møller scattering (elastic $e^- e^-$ scattering)**

Consider the Møller scattering in QED.

(a) {5 pts} **Tree level**

Use FORM to obtain the amplitude squared for unpolarized scattering at the tree level. It has the following structure:

$$\overline{|\mathcal{M}|^2} = \frac{f(s,t,u)}{t^2} + \frac{g(s,t,u)}{t\,u} + (t \leftrightarrow u), \qquad (2)$$

where invariant functions f and g depend on the Mandelstam variables s, t, u. Calculate the total cross section of the process.

(b) {3 pts} **A next-to-leading order contribution**

One of the higher-order contributions to Møller scattering is shown in Fig.3. Write down the unpolarized $|\mathcal{M}|^2$, in terms of the vaccum polarization $\Pi(p^2)$. It has the same structure as (2), with different f and g. What is here the contribution from the imaginary part of $\Pi(p^2)$?
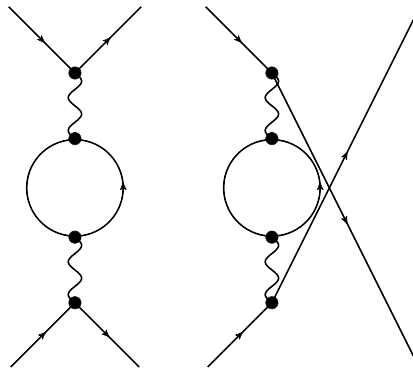

Figure 3: Diagrams that describe a part of the next-to-leading order contribution to Møller scattering.

4. **SU(3)$_C$ group**

   (a) {2 pts} **Properties of Gell-Mann matrices.**

   The λ matrices are normalized such that $\mathrm{Tr}(\lambda_a\lambda_b) = 2\delta_{ab}$ (for more properties see **Appendix, 1**).

   Check this for three combinations of matrices λ_a and λ_b. Calculate the structure constants $f_{156}, f_{345}, f_{458}$. Use $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$ to show that $\mathrm{Tr}(\lambda_c[\lambda_a, \lambda_b]) = 4if_{abc}$ and, by changing the order of the λ, that the structure constants f_{abc} are antisymmetric in the exchange of two indices.

   (b) {2 pts} **Color singlet hadrons.**

   Show that for quarks q_a ($a = 1, 2, 3 = r, g, b$) in the fundamental representation of color SU(3)$_C$, the combinations $\bar{q}_a q_a$ (mesons) and $\varepsilon_{abc} q_a q_b q_c$ (baryons) are color singlets, i.e. invariant under SU(3)$_C$ transformations: $q_a \to U_{ab}q_b$, $\bar{q}_b \to \bar{q}_b U^\dagger_{ba}$.

Appendix

1. **Structure Constants and Color Matrices**

   The 3×3 SU(3) generators, t_a, satisfy

   $$[t_a, t_b] = if_{abc}t_c \tag{3}$$

   where f_{abc} are the antisymmetric SU(3) structure constants with non-zero values given by

   $$f_{123} = 1, \quad f_{147} = \frac{1}{2}, \quad f_{246} = \frac{1}{2}, \quad f_{257} = \frac{1}{2}, \quad f_{367} = -\frac{1}{2}, \quad f_{678} = \frac{\sqrt{3}}{2} \tag{4}$$

   and $f_{156}, f_{345}, f_{458}$, can be found in Problem **4(b)**.

   A convenient representation of the $t_a = \lambda_a/2$ matrices is the one introduced by Gell-Mann in which

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

$$\tag{5}$$

The structure constants d_{abc} are defined according to

$$\{t_a, t_b\} = \frac{1}{3}\delta_{ab} + d_{abc}t_c . \tag{6}$$

The t_a matrices satisfy

$$t_a t_b = \frac{1}{2}\left[\frac{1}{3}\delta_{ab} + (d_{abc} + i f_{abc})t_c \right],$$

$$t_a^{ij} t_a^{kl} = \frac{1}{2}\left[\delta_{il}\delta_{jk} - \frac{1}{3}\delta_{ij}\delta_{kl} \right],$$

$$\mathrm{Tr}(t_a) = 0 ,$$

$$\mathrm{Tr}(t_a t_b) = \frac{1}{2}\delta_{ab} ,$$

$$\mathrm{Tr}(t_a t_b t_c) = \frac{1}{4}(d_{abc} + i f_{abc}) ,$$

$$\mathrm{Tr}(t_a t_b t_a t_c) = -\frac{1}{12}\delta_{bc} . \tag{7}$$

The structure constants satisfy the following Jacobi identities

$$f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbe}f_{ace} = 0 ,$$
$$f_{abe}d_{ecd} + f_{cbe}d_{ced} + f_{dbe}d_{ace} = 0 . \tag{8}$$

In addition,

$$f_{abc}f_{cde} = \frac{2}{3}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + (d_{ace}d_{bde} - d_{bce}d_{ade}) . \tag{9}$$

2. How to find and use FORM

   First, go https://www.nikhef.nl/~form/maindir/maindir.html.

   Here one can find all necessary documentation and installation files. For Linux users the complete instructions are given. If You are the Windows user and have not any desire to install Linux-type operation system as a second or as a virtual one, then You have 2 ways: install Cygwin or, in case of Windows 10, get e.g. "Ubuntu" app from Windows store. The second way is easier. You will get almost complete Ubuntu and simply follow the tutorials from above website. Anyway, be aware that You should have *gcc* and *gfortran* compilers inside the Cygwin or Ubuntu app to install and run FORM. In case of Cygwin You can choose them during its installation. In Ubuntu app You can simply install necessary compilers and apps from the Linux server using standard Linux commands, e.g.

   sudo apt-get install < ... >.

   One more advice is to install Microsoft Visual Studio Code. Being free and lightweight, it provides enough tools for convenient coding.