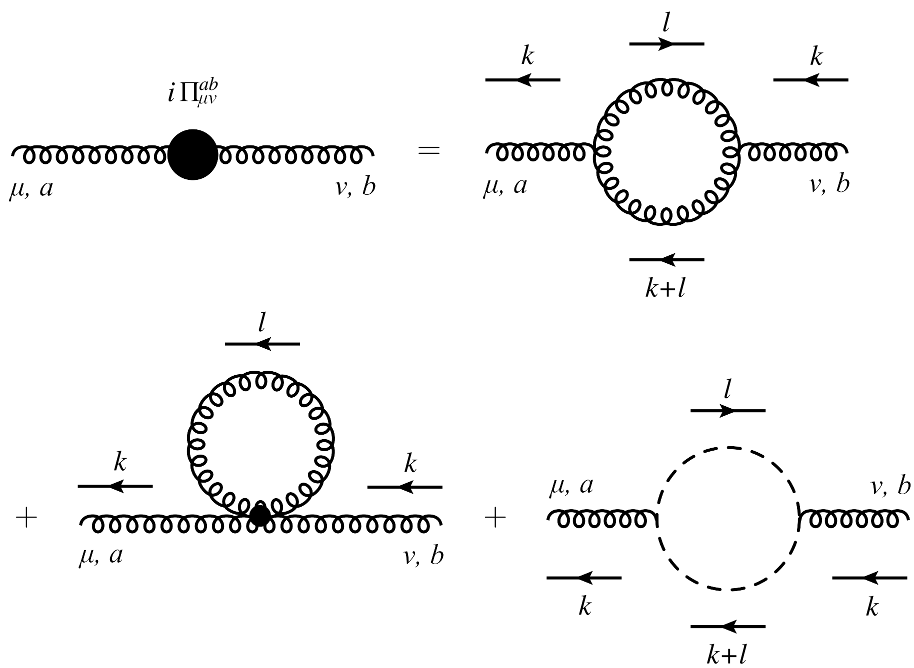
4 problems. Total number of points given is 27.
Grading: 20+ excellent, 15+ good, below 10 is incomplete.
Deadline: Thursday, December 5.

1. **Gluon self-energy**

   The one-loop contributions of the gluon self energy are given by these diagrams:



   The first two diagrams are due to the triple and quadratic gluon coupling, respectively. The third diagram is the Faddeev-Popov ghost contribution. The color indices a and b run from 1 to N_C.

   Assume the Feynman gauge for the gluon propagators:

$$\Delta^{ab}_{\mu\nu}(k) = -\delta^{ab}\frac{1}{k^2 + i\epsilon}g_{\mu\nu}$$

   For the other Feynman rules use the notes. Note the diagram with the ghost loop has an overall -1, just as for a fermion loop. Note also that the two diagrams with the gluon loop have the symmetry factor of 2.

The gauge invariance infers the transversality condition:

$$k_\mu \Pi^{\mu\nu}_{ab}(k) = 0\,, \tag{1}$$

leading to the following Lorentz stricture of the self-energy:

$$\Pi^{\mu\nu}_{ab}(k) = \delta_{ab}(k^2 g^{\mu\nu} - k^\mu k^\nu)\Pi(k^2)\,. \tag{2}$$

{9 pts} Richard Feynman had originally shown that the gluon loops alone (the first two diagrams) violate the above transversality condition, hence violate gauge invariance. He then introduced the ghost loop to restore the symmetry. Reproduce his result with the help of LoopTools.

2. **θ-term in QCD Lagrangian**

Consider the following additions to QED and QCD Lagrangians, the so-called "theta-term":

$$\begin{aligned} \Delta\mathcal{L}_{QED} &= \theta_{em}\frac{e^2}{32\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}, & (3)\\ \Delta\mathcal{L}_{QCD} &= \theta\frac{g^2}{32\pi^2}\mathcal{F}^a_{\mu\nu}\,\tilde{\mathcal{F}}^{a\,\mu\nu}, & (4) \end{aligned}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mathcal{F}^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, whereas

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \quad \tilde{\mathcal{F}}^{a\,\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\mathcal{F}^a_{\alpha\beta}$$

are the dual field-strength tensors, with ε being the Levy-Civita tensor.

(a) {2 pts} Why do these terms violate parity (P) and charge conjugation (C) symmetries?

(b) {2 pts} Show that the QED theta-term vanishes in the action:

$$\int d^4x \Delta\mathcal{L}_{QED} = 0.$$

(c) {1 pts} Can you say something on whether the action of the QCD theta-term vanishes or not?

3. **Furry's theorem**

   {1 pts} Prove that it is impossible to construct a nontrivial gauge-invariant interaction term for 3 photon fields, which conserves parity and charge conjugation. Namely, write down the coupling

$$\mathcal{L} = c_3 F^{\mu\nu} F_{\mu\alpha} F^{\alpha}_{\ \nu}$$

   where c_3 is a coupling constant, and show why this term vanishes.

   {2 pts} Generalize this proof for any odd coupling, i.e., $2n+1$ field tensors, with integer n. Write one or two sentences on how this is related to Furry's theorem.

4. **Constraints in Yang-Mills theory**

   Consider the SU(2) Yang-Mills theory, where two out of the three fields have the mass:

$$\mathcal{L}_{YM} = -\frac{1}{4}\sum_{a=1}^{3} \mathcal{F}^{a}_{\mu\nu}\,\mathcal{F}^{a\,\mu\nu} + \frac{1}{2}M^2[(A^1)^2 + (A^2)^2]$$

   {9 pts} Find all the constraints in this theory and classify them into first and second class.

   {1 pts} Perform the degrees-of-freedom counting.