Exercise sheet 7
Theoretical Physics 5 : WS 2019/2020
Lecturers : Prof. M. Vanderhaeghen, Dr. I. Danilkin

25.12.2019

Exercise 0.

How much time did it take to complete the task?

Exercise 1. (20 + [10 bonus] points) : Inversion operator

Under space inversion, a vector is transformed as

$$x^\mu \to x'^\mu = (a_P)^\mu{}_\nu x^\nu, \qquad \text{with} \qquad (a_P)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Find the explicit form of the corresponding inversion operator $P \equiv S(a_P)$.
Hint: Recall that for a transformation matrix a, the initial and transformed wave functions are related by $\psi'(x') = S(a)\psi(x)$ such that

$$S^{-1}(a)\gamma^\nu S(a) = a^\nu{}_\mu \gamma^\mu.$$

Bonus: Discuss whether P can be hermitian and can it be observable.

Exercise 2. (80 points) :
Dirac particle in a spherical potential well

Consider the Dirac equation

$$\left[\hat{\vec{\alpha}} \cdot \hat{\vec{p}}\, c + \hat{\beta} m_0 c^2\right] \psi(\vec{r}) = [E - V(r)]\psi(\vec{r}),$$

in a spherical potential well:

$$V(r) = \begin{cases} -V_0 & \text{for} \quad r \leq R, \\ 0 & \text{for} \quad r > R. \end{cases}$$

a) (*20 p.*) Show that

$$\hat{\vec{\alpha}} \cdot \hat{\vec{p}} = -i(\hat{\vec{\alpha}} \cdot \vec{e}_r)\left(\hbar \frac{\partial}{\partial r} + \frac{\hbar}{r} - \frac{\beta}{r}\hat{K}\right),$$

with $\hat{K} = \hat{\beta}\left(\hat{\vec{\Sigma}} \cdot \hat{\vec{L}} + \hbar\right)$ and $\vec{e}_r = \vec{r}/r$.
Hint: Use $\vec{\nabla} = \vec{e}_r(\vec{e}_r \cdot \vec{\nabla}) - \vec{e}_r \times (\vec{e}_r \times \vec{\nabla})$.

b) (*20 p.*) Use the ansatz

$$\psi(\vec{r}) = \begin{pmatrix} g(r)\chi_{\kappa,\mu}(\theta,\phi) \\ if(r)\chi_{-\kappa,\mu}(\theta,\phi) \end{pmatrix},$$

where χ are the eigenfunctions of the angular part:

$$\left(\hat{\vec{\sigma}} \cdot \hat{\vec{L}} + \hbar\right) \chi_{\kappa,\mu} = -\hbar\kappa\chi_{\kappa,\mu},$$
$$\left(\hat{\vec{\sigma}} \cdot \hat{\vec{L}} + \hbar\right) \chi_{-\kappa,\mu} = \hbar\kappa\chi_{-\kappa,\mu},$$

to find the differential equations for $u_1(r) = rg(r)$ and $u_2(r) = rf(r)$.

c) (*20 p.*) For $(\hbar kc)^2 \equiv (E+V_0)^2 - m_0^2c^4 > 0$ the general solution is given by:

$$u_1(r) = r\left[a_1 j_{l_\kappa}(kr) + a_2 y_{l_\kappa}(kr)\right],$$
$$u_2(r) = \frac{\kappa}{|\kappa|}\frac{\hbar ckr}{E+V_0+m_0c^2}\left[a_1 j_{l_{-\kappa}}(kr) + a_2 y_{l_{-\kappa}}(kr)\right],$$

where j_l and y_l are the spherical Bessel functions of the first and second kind.
For $(\hbar Kc)^2 \equiv m_0^2c^4 - (E+V_0)^2 > 0$ the general solution is given by:

$$u_1(r) = r\sqrt{\frac{2Kr}{\pi}}\left[b_1 K_{l_\kappa+1/2}(Kr) + b_2 I_{l_\kappa+1/2}(Kr)\right],$$
$$u_2(r) = \frac{\hbar cKr}{E+V_0+m_0c^2}\sqrt{\frac{2Kr}{\pi}}\left[-b_1 K_{l_{-\kappa}+1/2}(Kr) + b_2 I_{l_{-\kappa}+1/2}(Kr)\right],$$

where $K_{l+1/2}$ and $I_{l+1/2}$ are the modified Bessel functions. Furthermore, it is

$$l_\kappa = \begin{cases} \kappa & \text{for} \quad \kappa > 0, \\ -\kappa - 1 & \text{for} \quad \kappa < 0. \end{cases}$$

and

$$l_{-\kappa} = \begin{cases} -\kappa & \text{for} \quad -\kappa > 0, \\ \kappa - 1 & \text{for} \quad -\kappa < 0. \end{cases}$$

Determine the bound states, for which $E > -V_0 + m_0c^2$, $-m_0c^2 < E < m_0c^2$.

d) (*20 p.*) Exploiting the continuity condition at $r = R$, derive the following relation for s-states ($\kappa = -1$):

$$\frac{kR\sin(kR)}{\sin(kR) - kR\cos(kR)} = \frac{k}{K}\frac{e^{-KR}}{e^{-KR}\left(1+\frac{1}{KR}\right)}\frac{E+m_0c^2}{E+V_0+m_0c^2}.$$

Rewrite the above equation into:

$$\tan\left(\frac{R}{\hbar c}\sqrt{(E+V_0)^2 - m_0^2c^4}\right)\sqrt{\frac{E+V_0+m_0c^2}{E+V_0-m_0c^2}}\times$$
$$\left\{\frac{\hbar c}{R}\left[\frac{1}{E+m_0c^2} - \frac{1}{E+V_0+m_0c^2}\right] - \sqrt{\frac{m_0c^2-E}{m_0c^2+E}}\right\} = 1.$$

These equations relate the energy eigenvalues of the s-states and the properties of the spherical potential well.

Hint: If necessary, you can use the following Rayleigh formulas for (spherical) Bessel functions:

$$\begin{aligned}
j_n(x) &= (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sin(x)}{x}\right), \\
y_n(x) &= -(-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\cos(x)}{x}\right), \\
i_n(x) &= x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{\sinh(x)}{x}\right), \\
k_n(x) &= (-1)^n x^n \left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \left(\frac{e^{-x}}{x}\right),
\end{aligned}$$

with

$$i_n(x) = \sqrt{\frac{\pi}{2x}} I_{n+1/2}(x) \qquad \text{and} \qquad k_n(x) = \sqrt{\frac{\pi}{2x}} K_{n+1/2}(x).$$