Exercise sheet 6
Theoretical Physics 5 : WS 2019/2020
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Exercise 0.

How much time did it take to complete the task?

Exercise 1. (45 points) : Dirac matrices gymnastics

Without using an explicit representation for the Dirac matrices, show that:

a) *(5 p.)* $\gamma_\mu \gamma^\mu = 4$;

b) *(5 p.)* $\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4 g^{\mu\nu}$;

c) *(5 p.)* $\mathrm{Tr}[\not{a}\,\not{b}\,\not{c}\,\not{d}] = 4\,(a_\mu b^\mu c_\nu d^\nu - a_\mu c^\mu b_\nu d^\nu + a_\mu d^\mu b_\nu c^\nu)$, where $\not{a} \equiv \gamma^\mu a_\mu$;

d) *(5 p.)* $\gamma_5^2 = \mathbb{1}$, with $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$;

e) *(5 p.)* $\{\gamma_5, \gamma^\mu\} = 0$;

f) *(5 p.)* $\mathrm{Tr}[\gamma_5] = 0$;

g) *(5 p.)* $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma_5] = 0$;

h) *(5 p.)* $\mathrm{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_5] = 4i\,\varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon_{0123} = +1$;

i) *(5 p.)* $\mathrm{Tr}[\gamma^{\mu_1} \cdots \gamma^{\mu_n}] = 0$ if n is odd.

Exercise 2. (10 + [20 bonus] points) : Lorentz transformation identity

Verify that for an infinitesimal Lorentz transformation $S = \mathbb{1} + \frac{1}{8}[\gamma_\mu, \gamma_\nu](\delta\omega)^{\mu\nu}$:

$$S^{-1} = \gamma_0 S^\dagger \gamma_0.$$

[*Bonus*] Verify that for arbitrary proper Lorentz transformation.

Exercise 3. (15 points) : Weyl representation

In the standard Dirac representation, Dirac matrices have the form

$$\gamma_s^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \text{and} \qquad \vec{\gamma}_s = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix},$$

while in the so-called *Weyl representation*, they have the form

$$\gamma_W^\mu \quad = \quad \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix},$$

with $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$ and $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$.

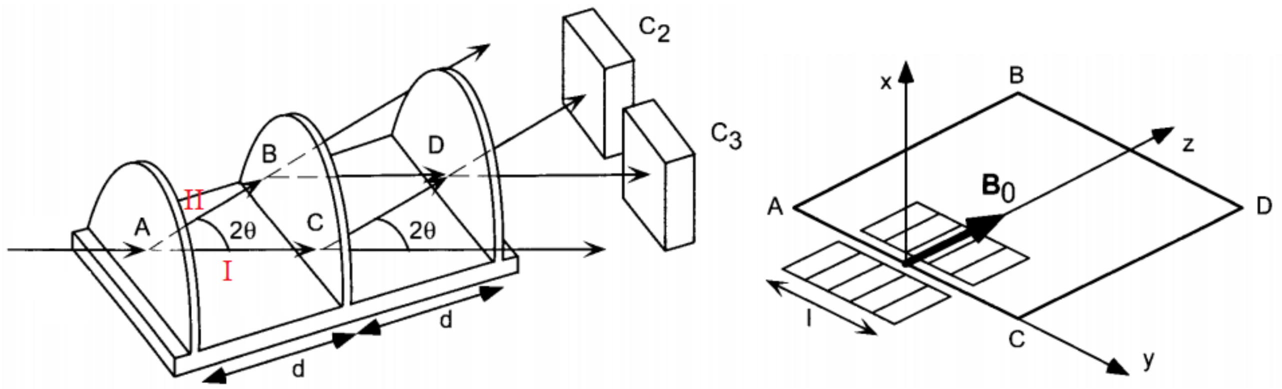
a) *(5 p.)* Write down a unitary matrix S connecting both representations $\gamma_s^\mu = S\gamma_W^\mu S^{-1}$.

b) *(5 p.)* Write the γ_5 matrix in both representations.

c) *(5 p.)* In the Weyl representation with $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, show that the bispinors ψ_L and ψ_R are independent for massless particles, and write down the eigenstates of the chirality operator γ_5 with their corresponding eigenvalues.

Exercise 4. (30 + [10 bonus] points) : Neutron Interferometry

Let us consider a neutron interferometer which is shown in the figure.



The (approximately) monochromatic neutron beam is passing through and being split by three parallel silicon strips. The outgoing neutron fluxes are measured by detectors C_2 and C_3.

The incident neutron beam is forming the Bragg angle θ with respect to the perpendicular direction to the crystal plate so that it is split by the crystal into two symmetric outgoing waves. A plain wave $\psi_0 = e^{i(-Et+\vec{p}\cdot\vec{r})/\hbar}$, where E is the energy of the neutrons and $\vec{p}$ their momentum, is split into

$$\psi_{\mathrm{I}} = \alpha\, e^{i(-Et+\vec{p}\cdot\vec{r})/\hbar} \qquad \text{and} \qquad \psi_{\mathrm{II}} = \beta\, e^{i(-Et+\vec{p}\,'\cdot\vec{r})/\hbar}$$

on each silicon strip. Since the scattering is elastic, $|\vec{p}\,'| = |\vec{p}|$. The complex amplitudes α and β can be parameterized as $\alpha = \cos\chi$ and $\beta = i\sin\chi$, with χ being real. The transmission and reflection coefficients are $T = |\alpha|^2$ and $R = |\beta|^2$, respectively. We assume the spin state is not changed by the interaction with a silicon panel.

We can create a phase difference δ of the wave propagating along AC with respect to the wave propagating along AB by placing along AC a magnet of length l which produces a constant uniform magnetic field $\vec{B}_0$ along the z-axis (see right panel of the figure). This results in a difference of the neutron fluxes between the counters C_2 and C_3.

We assume space and spin variables can be factorized and the spin of the incident neutrons is along the x-axis

$$\chi(t=0) = \chi_{\uparrow}^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

Assume $l = 2.8$ cm and the de Broglie wavelength is fixed to $\lambda = 1.445$ Å. Neutron mass is $m_n = 1.675 \times 10^{-27}$ kg, $\mu_0 = -9.65 \times 10^{-27}$ J T^{-1}.

a) (*7 p.*) Consider the neutron inside the magnetic field.
Recall that the neutrons are spin-1/2 particles, which can be described by the Pauli equation. Write the interaction Hamiltonian of the spin with the external magnetic field. How does the neutron spin state in the magnet evolve in time?
Show that the expectation value of the intrinsic magnetic moment operator $\vec{\mu} = \mu_0\vec{\sigma}$ in the magnet is

$$\langle\vec{\mu}\rangle = \mu_0\left(\cos(\omega t)\vec{e}_x + \sin(\omega t)\vec{e}_y\right),$$

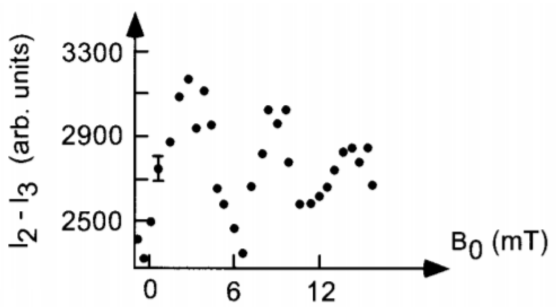
with $\omega = -2\mu_0 B_0/\hbar$.

b) (*8 p.*) Consider the neutron after it leaves the magnet.
It happens after the time $t_o = l/v = m_n l\lambda/(2\pi\hbar)$. What is the probability of measuring $+\mu_0$ value of the x component of the neutron magnetic moment?
For which values b_n of the field B_0 is this probability equal to 1?
Show that the phase $\delta \equiv \omega t_o/2 = n\pi$ corresponds to a rotation of the spin vector by $2n\pi$ around the z-axis by Larmor precession.

c) (*10 p.*) Consider the neutron propagation in the interferometer.
What is the state of the neutrons arriving on the counters C_2 and C_3?

d) (*5 p.*) Find the neutron fluxes I_2 and I_3 on C_2 and C_3 in terms of the transmission and reflection coefficients and the phase $\delta \equiv \omega t_o/2$.

e) (*10 p.*)[*Bonus*] The experimental data on $I_2 - I_3$ as a function of the applied magnetic field is shown in the figure below. The extracted distance between two maxima is $\Delta B = 64 \pm 2$ mT.



Compare this result with the theoretical values of b_n you have obtained.
How does this prove that rotation of a spin-1/2 particle by $2\pi n$, with n being **odd**, changes sign of its state vector?