Exercise sheet 4
Theoretical Physics 5 : WS 2019/2020
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04.11.2019

Exercise 0.

How much time did it take to complete the task?

Exercise 1. (40 points) : Real Klein-Gordon field

Using the normal mode expansion of the real Klein-Gordon field

$$\phi(\vec{x},t) \;=\; \sum_{\vec{k}} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \left[a(\vec{k})\, e^{-ik.x} + a^\dagger(\vec{k})\, e^{ik.x}\right]$$

and the equal-time commutation relations

$$\begin{aligned}
&[\phi(\vec{x},t), \phi(\vec{x}',t)] = 0,\\
&[\dot{\phi}(\vec{x},t), \dot{\phi}(\vec{x}',t)] = 0,\\
&[\phi(\vec{x},t), \dot{\phi}(\vec{x}',t)] = i\hbar\, c^2\, \delta^{(3)}(\vec{x}-\vec{x}'),
\end{aligned}$$

Show that:

a) (*20 p.*) the creation and annihilation operators satisfy the following commutation relations

$$\begin{aligned}
&[a(\vec{k}), a(\vec{k}')] = 0,\\
&[a^\dagger(\vec{k}), a^\dagger(\vec{k}')] = 0,\\
&[a(\vec{k}), a^\dagger(\vec{k}')] = \delta_{\vec{k},\vec{k}'};
\end{aligned}$$

b) (*10 p.*) the Hamiltonian $H = \int d^3x\, \frac{1}{2}\left[\frac{1}{c^2}\,\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + \mu^2\phi^2\right]$ takes the form

$$H = \sum_{\vec{k}} \hbar\omega_k \left(a^\dagger(\vec{k})a(\vec{k}) + \frac{1}{2}\right);$$

c) (*10 p.*) the momentum $\vec{P} = -\int d^3x\, \frac{1}{c^2}\,\dot{\phi}\,\vec{\nabla}\phi$ takes the form

$$\vec{P} = \sum_{\vec{k}} \hbar\vec{k}\; a^\dagger(\vec{k})a(\vec{k}).$$

Exercise 2. (60 points) : Complex Klein-Gordon field

The complex Klein-Gordon field is used to describe charged bosons. Its Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - \mu^2 \phi^\dagger \phi, \tag{1}$$

where the field ϕ has the following normal mode expansion

$$\phi(\vec{x},t) = \sum_{\vec{k}} \left(\frac{\hbar c^2}{2\omega_k L^3}\right)^{1/2} \left[a(\vec{k})\, e^{-ik\cdot x} + b^\dagger(\vec{k})\, e^{ik\cdot x}\right]$$

and satifies the equal-time commutation relations

$$\begin{aligned} [\phi(\vec{x},t), \Pi_\phi(\vec{x}',t)] &= i\hbar\, \delta^{(3)}(\vec{x}-\vec{x}'), \\ \left[\phi^\dagger(\vec{x},t), \Pi_{\phi^\dagger}(\vec{x}',t)\right] &= i\hbar\, \delta^{(3)}(\vec{x}-\vec{x}'), \end{aligned}$$

all other commutators vanishing. In the following, you can conveniently consider the fields ϕ and $\phi^\dagger$ as independent.

a) (*15 p.*) Show that (1) is equivalent to the Lagrangian of two independent real scalar fields with same mass and satisfying the standard equal-time commutation relations.
Hint: Decompose the complex field in real components $\phi = \frac{1}{\sqrt{2}}\,(\phi_1 + i\phi_2)$.

b) (*15 p.*) Write down the conjugate momentum fields Π_ϕ and $\Pi_{\phi^\dagger}$ in terms of ϕ and $\phi^\dagger$, and derive the equal-time commutation relations of a, $a^\dagger$, b and $b^\dagger$.

c) (*15 p.*) Show that (1) is invariant under any global phase transformation of the field $\phi \to \phi' = e^{-i\alpha}\phi$ with α real. Write down the associated conserved Noether current J^μ and express the conserved charge $Q = \int d^3x\, J^0$ in terms of creation and annihilation operators.

d) (*15 p.*) Compute the commutators $[Q,\phi]$ and $[Q,\phi^\dagger]$. Using these commutators and the eigenstates $|q\rangle$ of the charge operator Q, show that the field operators ϕ and $\phi^\dagger$ modify the charge of the system. How would you interpret the operators a, $a^\dagger$, b and $b^\dagger$?

[Bonus] Exercise 3. (30 points) : Pionic atoms

A pionic atom is formed when a negative pion π^-, which is a spin-0 boson, is stopped in matter and is captured by an atom. The incident pion slows down by successive electromagnetic interactions with the electrons and nuclei, and when it reaches the typical velocity of atomic electrons, the pion is captured by ejecting a bound electron from its Bohr orbit. Let us approximate the potential between the nucleus and the pion by a square-well $V = -V_0$ for $r \le R$ and $V = 0$ for $r > R$, where R is the nucleus radius.

a) (*10 p.*) Using the minimal substitution $p_\mu \to p_\mu - \frac{e}{c}A_\mu$, with $A_\mu = (V,\vec{0})$, show that the Klein-Gordon equation leads to the following radial equation for the field

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right] u(r) = 0,$$

where $k^2 = \frac{1}{\hbar^2 c^2}\left[(\epsilon - eV)^2 - m_\pi^2 c^4\right]$ with ϵ the energy of the pion.
Hint: Use the Klein-Gordon field in the following factorized form $\phi(\vec{x},t) = u(r)\, Y_{lm}(\Omega)\, e^{-\frac{i}{\hbar}\epsilon t}$ with $Y_{lm}(\Omega)$ the standard spherical harmonic functions.

b) (*10 p.*) Since for a bound state we have $k^2 > 0$ for $r \leq R$ and $k^2 < 0$ for $r > R$, solve the equation for an s-state ($l = 0$) in both regions.
Hint: Use the *Ansatz* $u(r) = v(r)/r$.

c) (*10 p.*) Match the solutions by imposing equal logarithmic derivatives $\frac{1}{u_i}\frac{du_i}{dr} = \frac{1}{u_o}\frac{du_o}{dr}$ at $r = R$, and show that this amounts to solve the transcendental equation

$$k_i \cot(k_i R) = -k_o,$$

where $k_i^2 = \frac{1}{\hbar^2 c^2}\left[(\epsilon + eV_0)^2 - m_\pi^2 c^4\right]$ and $k_o^2 = \frac{1}{\hbar^2 c^2}\left(m_\pi^2 c^4 - \epsilon^2\right)$.