# Exercise sheet 5
# Theoretical Physics 5 : WS 2019/2020
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### Exercise 0.

How much time did it take to complete the task?

## Exercise 1. (60 points) : Dirac particle in a scalar potential

Consider a Dirac particle travelling along the $z$-axis and subject to the scalar square-well potential

$$V(z) = \begin{cases} 0 & \text{when} \quad z < -a/2 & \text{(region I)} \\ V_0 & \text{when} \quad -a/2 \leq z \leq a/2 & \text{(region II)} \\ 0 & \text{when} \quad a/2 < z & \text{(region III)} \end{cases}$$

where $a > 0$ and $V_0 < 0$. In regions I and III, the eigenvalue Dirac equation takes the form

$$\left(\vec{\alpha} \cdot \hat{\vec{p}} c + \beta m_0 c^2\right) \psi = E\psi,$$

while in region II, it has the form

$$\left[\vec{\alpha} \cdot \hat{\vec{p}} c + \beta(m_0 c^2 + V_0)\right] \psi = E\psi.$$

In this second region one can consider that due to the potential, the particle has now an effective mass $m_{\text{eff}} = m_0 + V_0/c^2$.

a) (*10 p.*) Write down the general solution $\psi(z)$ in the three regions with the spin in the $z$-direction.
*Hints*: A plane-wave solution with momentum $\vec{p}$, mass $m$ and spin label $s$ can be written as

$$u(\vec{p}, s) = A \begin{pmatrix} \chi_s \\ \frac{c\ \vec{\sigma}\cdot\vec{p}}{E+mc^2} \chi_s \end{pmatrix} e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - Et)},$$

where $A$ is some complex number and $\chi_s$ a two-component spinor. For the spin projection along the $z$-axis, $\chi_s$ is an eigenstate of the Pauli matrix $\sigma_3$. Do not forget that plane waves can travel in both directions. Remember that the matrices $\vec{\alpha}$ and $\beta$ in standard representation are given by

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \text{and} \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

b) (*20 p.*) Impose the continuity condition at $z = \pm a/2$. Express the coefficient of the incoming plane wave of region I in terms of the plane waves of region III. Define for convenience the dimensionless quantity

$$\gamma \equiv \frac{k_1 c}{E + m_0 c^2} \frac{E + m_{\text{eff}} c^2}{k_2 c},$$

where $k_1$ is the momentum in regions I and III, and $k_2$ is the momentum in region II.

c) (*30 p.*) Consider the special case $|m_{\text{eff}} c^2| < |E| < m_0 c^2$ corresponding to bound states. Show that these states satisfy

$$k_2 \cot\left(\frac{k_2 a}{\hbar}\right) = -\left(\frac{m_0 V_0}{\kappa_1} + \kappa_1\right),$$

where $\kappa_1 = -ik_1$.
*Hints*: Show that in the case considered, there could be neither an incoming wave in region I nor an outgoing wave in region III. Show then that continuity requires

$$\text{Im}\left(\frac{1+\gamma}{1-\gamma}\, e^{-\frac{i}{\hbar} k_2 a}\right) = 0,$$

and that $\gamma = i\Gamma$ is imaginary leading then to

$$\cot\left(\frac{k_2 a}{\hbar}\right) = \frac{1 - \Gamma^2}{2\Gamma}.$$

# Exercise 2. (15 points) : Helicity operator

The helicity operator is defined as $\hat{\Lambda} = \vec{S} \cdot \frac{\vec{p}}{|\vec{p}|}$. Derive the eigenvalues of the helicity operator for a particle with momentum $p = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$.

# Exercise 3. (25 points) : Electrons in constant magnetic field

To study the behavior of electrons in a constant magnetic field, we have to solve the stationary-state Dirac equation

$$\left(-i\hbar c\, \vec{\alpha} \cdot \vec{D} + \beta m_0 c^2\right) \psi = E\psi$$

where we have used minimal substitution $\vec{\nabla} \to \vec{D} \equiv \vec{\nabla} - ie\vec{A}/\hbar c$.

a) (*10 p.*) Verify that

$$\left(\vec{\alpha} \cdot \vec{D}\right)^2 = \vec{D}^2 \mathbb{1} + \frac{e}{\hbar c} \vec{\Sigma} \cdot \vec{B}, \qquad \text{where} \qquad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}.$$

b) (*15 p.*) For the particular case $\vec{A} = (0, xB, 0)$ and by considering solutions of the form $\psi = e^{i(p_y y + p_z z)/\hbar} u(x)$, show that the energy eigenvalues $E$ of a relativistic electron in constant magnetic induction $\vec{B}$ are given by

$$E^2 = m_0^2 c^4 + p_z^2 c^2 + (2n+1)|eB|\hbar c \pm eB\hbar c, \qquad n \in \mathbb{N}.$$

*Hint*: Remember that the eigenvalues of the harmonic oscillator operator $-\partial_x^2 + \omega^2 x^2$ are $(2n+1)|\omega|$ with $n \in \mathbb{N}$.