

Vollständigkeit

1. Endliches Intervall $I = [a, b]$

$$f(x) = \sum_i a_i u_i(x) \quad \text{mit} \quad a_i = \langle u_i, f \rangle = \int_a^b dx u_i^*(x) f(x)$$

Vollständigkeit

$$\delta(x - x') = \sum_i u_i(x) u_i^*(x')$$

Dann gilt

$$\begin{aligned} f(x) &= \int_a^b dx' \delta(x - x') f(x') \\ &= \int_a^b dx' \sum_i u_i(x) u_i^*(x') f(x') \\ &= \sum_i u_i(x) \underbrace{\int_a^b dx' u_i^*(x') f(x')}_{= a_i} \\ &= \sum_i a_i u_i(x) \end{aligned}$$

2. Unendliches Intervall \mathbb{R}

$$f(x) = \int_{-\infty}^{\infty} dk a(k) \frac{e^{ikx}}{\sqrt{2\pi}} \quad \text{mit} \quad a(k) = \int_{-\infty}^{\infty} dx \frac{e^{-ikx}}{\sqrt{2\pi}} f(x)$$

Vollständigkeit

$$\delta(x - x') = \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\sqrt{2\pi}} \frac{e^{-ikx'}}{\sqrt{2\pi}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(x-x')}$$

Dann gilt

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} dx' \delta(x - x') f(x') \\ &= \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk \frac{1}{2\pi} e^{ik(x-x')} f(x') \\ &= \int_{-\infty}^{\infty} dk \frac{e^{ikx}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} dx' \frac{e^{-ikx'}}{\sqrt{2\pi}} f(x')}_{= a(k)} \\ &= \int_{-\infty}^{\infty} dk a(k) \frac{e^{ikx}}{\sqrt{2\pi}} \end{aligned}$$