Examples Sheet 2

Symmetries in Physics

Winter 2019/20

Lecturer: PD Dr. G. von Hippel

1. *Properties of the structure constants* (5 P.)

   Let $\mathfrak{g}$ be a Lie algebra with structure constants f^a_{bc}. Show that the structure constants satisfy

   1. $f^a_{bc} = -f^a_{cb}$
   2. $f^a_{bc} f^c_{de} + f^a_{dc} f^c_{ea} + f^a_{ec} f^c_{bd} = 0$

2. *Low-dimensional Lie algebras* (10 P.)

   1. Determine all Lie algebras of dimension 1.
   2. Show that all Lie algebras of dimension 2 can be reduced to either $[L_1, L_2] = 0$ or $[L_1, L_2] = L_2$ by a change of basis (with all other Lie brackets following from this or being trivially zero).

3. *Disconnectedness of GL($n, \mathbb{R}$)* (5 P.)

   Show that GL($n, \mathbb{R}$)$= \{M \in \mathbb{R}^{n \times n} | \det(M) \neq 0\}$ is not connected. [*Hint:* Consider the determinant and find two points such that it must vanish at some point along any path joining them.]

4. *Adjoint representation* (10 P.)

   Let $\mathfrak{g}$ be a Lie algebra with structure constants f^a_{bc}. Define matrices T_b by $(T_b)^a_c = f^a_{bc}$. Show that the T_b satisfy the commutation relations $[T_b, T_c] = f^a_{bc} T_a$ defining $\mathfrak{g}$.

5. *Angular momentum algebra* (10 P.)

   1. Show that $\mathbb{R}^3$ with the cross product as the Lie bracket forms a Lie algebra.
   2. Determine the structure constants of this Lie algebra in the canonical basis.
   3. Find the matrices of the adjoint representation (cf. previous question). Where have you seen these matrices before?