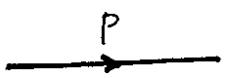
RENORMALIZATION IN QCD

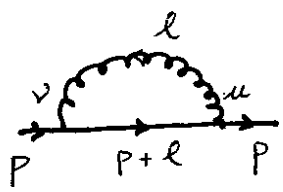
⇒ LOOP DIAGRAMS

↳ LOOP DIAGRAMS CORRESPOND WITH QUANTUM CORRECTIONS

• QUARK SELF-ENERGY



FREE QUARK PROPAGATOR $\dfrac{i(\not p + m)}{p^2 - m^2}$



GLUON 'DRESSING' OF THE QUARK
(SELF ENERGY) MODIFIES ITS PROPERTIES
$\Sigma(p)$

$$-i\Sigma(p) \equiv \int \frac{d^4\ell}{(2\pi)^4} \, \frac{i}{\ell^2}\left[-g_{\mu\nu} + (1-\xi)\, \ell_\mu \ell_\nu / \ell^2\right]$$

$$\cdot \frac{1}{3} \underbrace{\mathrm{Tr}\left\{\frac{\lambda_a}{2}\frac{\lambda_a}{2}\right\}}_{\frac{1}{2}\cdot \underbrace{\delta_{aa}}_{=8}} \cdot \left[-ig\gamma^\mu\right] \frac{i(\not p + \not \ell + m)}{\left[(p+\ell)^2 - m^2\right]} \left[-ig\gamma^\nu\right]$$

AVERAGE OVER 3 COLORS

COLOR FACTOR : $\frac{4}{3}$

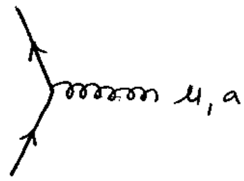
FOR $\ell \gg$ PART OF LOOP

$$-i\Sigma(p) \longrightarrow \int d^4\ell \, \frac{1}{\ell^2} \, \frac{\ell}{\ell^2} \Rightarrow \text{LINEARLY DIVERGENT}$$

• QUARK – GLUON VERTEX

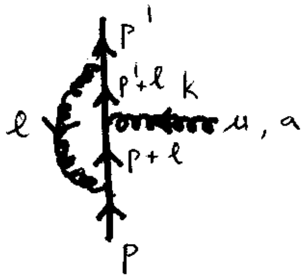
↳ LOWEST ORDER

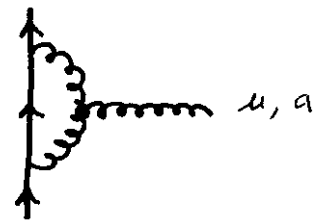
$O(g)$



$$-ig\,\frac{\lambda_a}{2}\,\gamma^\mu$$

↳ $O(g^3)$





$$-ig\,\Lambda_a^{(1)\mu} \qquad\qquad -ig\,\Lambda_a^{(2)\mu}$$

$$-ig\,\Lambda_a^{(1)\mu} = \int\frac{d^4\ell}{(2\pi)^4}\left[-ig\,\frac{\lambda_b}{2}\gamma^\beta\right]\frac{i(\not p'+\not\ell+m)}{(p'+\ell)^2-m^2}\left[-ig\,\frac{\lambda_a}{2}\gamma^\mu\right]$$

$$\frac{i(\not p+\not\ell+m)}{(p+\ell)^2-m^2}\left[-ig\,\frac{\lambda_b}{2}\gamma^\alpha\right]$$

$$\cdot\,\frac{i}{\ell^2}\left[-g_{\alpha\beta}+(1-\xi)\,\ell_\alpha\ell_\beta/\ell^2\right]$$

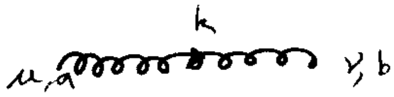
$$= -ig\,(ig^2)\,\frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2}$$

$$\cdot\int\frac{d^4\ell}{(2\pi)^4}\,\frac{\gamma^\beta(\not p'+\not\ell+m)\,\gamma^\mu(\not p+\not\ell+m)\,\gamma^\alpha}{\left[(p'+\ell)^2-m^2\right]\left[(p+\ell)^2-m^2\right]\,\ell^2}\left\{-g_{\alpha\beta}+(1-\xi)\frac{\ell_\alpha\ell_\beta}{\ell^2}\right\}$$

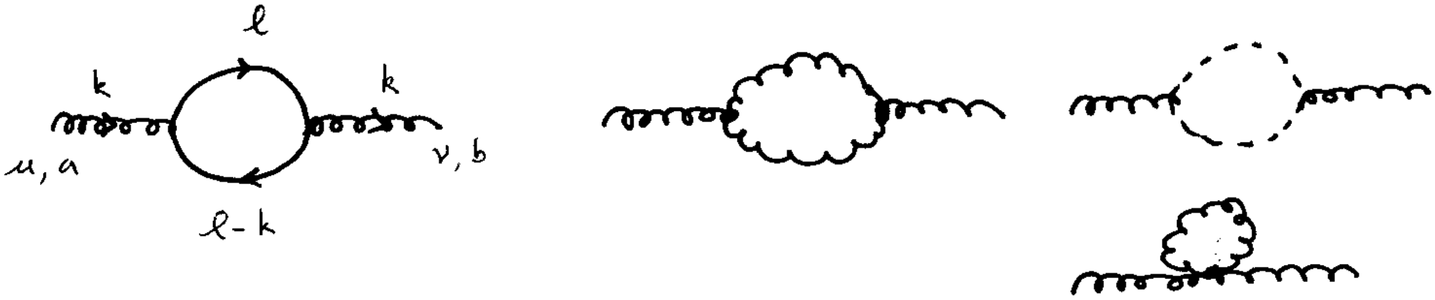
$$\xrightarrow[\ell\gg]{}\int d^4\ell\,\frac{\ell^2}{\ell^6}\sim\int d^4\ell\,\frac{1}{\ell^4}$$

LOGARITHMIC DIVERGENCE

• GLUON SELF-ENERGY



$$\delta_{ab} \frac{i}{k^2} \left[-g_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right]$$



$$-i\Pi^{\mu\nu}_{ab}(k)_{\substack{\text{QUARK}\\ \text{LOOP}}} = \int \frac{d^4\ell}{(2\pi)^4} \underset{\substack{\uparrow \\ \text{FERMION} \\ \text{LOOP}}}{(-1)} \operatorname{Tr}\left\{ \left[-ig \frac{\lambda_b}{2} \gamma^\nu \right] \frac{i(\ell\!\!/ + m)}{\ell^2 - m^2} \left[-ig \frac{\lambda_a}{2} \gamma^\mu \right] \cdot \frac{i(\ell\!\!/ - k\!\!/ + m)}{(\ell-k)^2 - m^2} \right\}$$

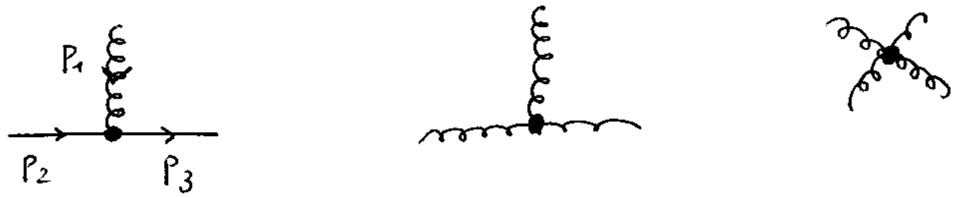
$$= -\underbrace{\operatorname{Tr}\left\{ \frac{\lambda_b}{2} \frac{\lambda_a}{2} \right\}}_{\frac{1}{2}\delta_{ab}} \cdot \int \frac{d^4\ell}{(2\pi)^4} \frac{\operatorname{Tr}\left\{ \gamma^\nu (\ell\!\!/ + m) \gamma^\mu (\ell\!\!/ - k\!\!/ + m) \right\}}{\left[\ell^2 - m^2\right]\left[(\ell-k)^2 - m^2\right]}$$

$$\xrightarrow[\ell \gg]{} \int d^4\ell \, \frac{\ell^2}{\ell^4} \sim \int d^4\ell \, \frac{1}{\ell^2}$$

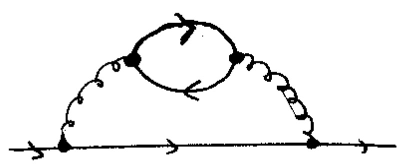
QUADRATIC DIVERGENCE

⇒ DEGREE OF DIVERGENCE OF ARBITRARY DIAGRAM

FOR EVERY VERTEX OF TYPE i



e.g. ENTERS



- EVERY MOMENTUM $\curvearrowright$ $\displaystyle\int \frac{d^4 p_i}{(2\pi)^4}$

- EVERY FERMION PROPAGATOR CONNECTING 2 VERTICES LEADS TO A "MOMENTUM" POWER AT EACH VERTEX

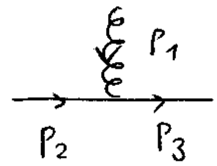
$$\frac{1}{2} \int d^4 p_i \quad \frac{\not{p}_i + m}{p_i^2 - m^2} \longrightarrow \frac{3}{2}$$

BECAUSE PROPAGATOR CONNECTS 2 VERTICES

- EVERY BOSON PROPAGATOR LEADS TO A "MOMENTUM" POWER AT EACH VERTEX

$$\frac{1}{2} \cdot \int d^4 p_i \cdot \frac{1}{p_i^2 - m^2} \left(- g_{\mu\nu} + \frac{p_{i\mu} p_{i\nu}}{p_i^2} (1 - \xi) \right) \longrightarrow 1$$

- FOR EVERY VERTEX : THERE IS 1 ENERGY - MOMENTUM CONSERVATION



$\rightarrow \; p_1 + p_2 = p_3$

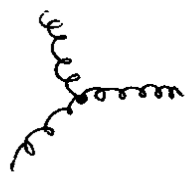
$\delta^4 \left(\sum_i p_i \right)$ $\rightarrow$ REDUCING MOMENTUM POWER COUNTING BY 4 AT EACH VERTEX.

$\Downarrow$

-4

- IF WE HAVE DERIVATIVE INTERACTIONS

e.g.



$d_i = 1$

$d_i = \#$ DERIVATIVES IN INTERACTION $\mathcal{L}$.

$\Downarrow$

LEADS TO MOMENTUM POWER d_i AT EACH VERTEX.

$\therefore$ DEGREE OF DIVERGENCE (DD) WHEN SUMMING OVER ALL VERTICES

$$DD = \frac{\# \text{ MOMENTUM POWERS IN NUMERATOR}}{\# \text{ MOMENTUM POWERS IN DENOMINATOR}}$$

$$DD = \sum_i n_i \cdot \left(d_i + \frac{3}{2} f_i + b_i - 4 \right)$$

SUM OVER ALL VERTICES OF TYPE i

VERTICES OF TYPE i

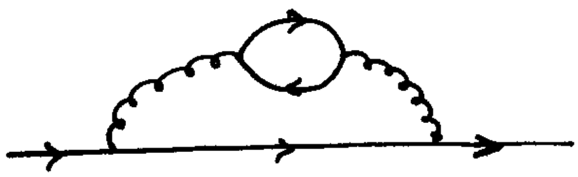
DERIVATIVES IN $\mathcal{L}$ AT VERTEX i

FERMION LINES ENTERING VERTEX i

BOSON LINES ENTERING VERTEX i

ENERGY - MOMENTUM CONSERV.

- CORRECT FOR EXTERNAL LINES



→ BECAUSE FOR EXTERNAL LINES : NO PROPAGATOR

→ WE SUBTRACTED ONE ENERGY-MOMENTUM CONSERVATION TOO MUCH, BECAUSE ONE CORRESPONDS WITH OVERALL ENERGY-MOMENTUM CONSERVATION

↓

LEADS TO NO REDUCTION IN MOMENTUM POWER.

$$\therefore \quad \boxed{DD = \sum_i n_i \cdot \delta_i + \left(4 - \frac{3}{2} N_f - N_b\right)}$$

N_f : # EXTERNAL FERMION LINES

N_b : # " BOSON "

δ_i : INDEX OF DIVERGENCE OF $\mathcal{L}_i$

$$\delta_i \equiv d_i + \frac{3}{2} f_i + b_i - 4$$

- IF $\delta_i > 0$

  ↳ MORE & MORE DIVERGENCES APPEAR WHEN CALCULATING HIGHER LOOPS

  $\Downarrow$

  NON - RENORMALIZABLE THEORY

- IF $\delta_i < 0$

  ↳ THEORY BECOMES MORE CONVERGENT WHEN CALCULATING HIGHER LOOPS

  SUCH THEORIES ARE TRIVIAL

  $\Downarrow$

  SUPER - RENORMALIZABLE THEORY

- IF $\delta_i = 0$

  ↳ NO NEW DIVERGENCES APPEAR WHEN CALCULATING HIGHER LOOP DIAGRAMS COMPARED WITH 1 LOOP DIAGRAMS

  $\Downarrow$

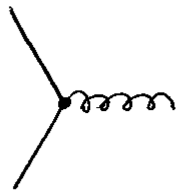
  RENORMALIZABLE THEORY

'PHYSICAL' THEORIES

QED, QCD, STANDARD ELECTROWEAK GAUGE THEORY

- e.g. FOR QCD

$\mathcal{L}_{qqg}$

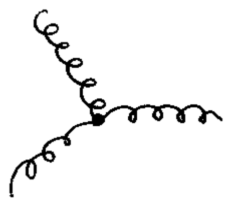


$f_i = 2 \quad , \quad b_i = 1 \quad , \quad d_i = 0$

$$\delta_i = \frac{3}{2} f_i + b_i + d_i - 4$$

$$\longrightarrow \delta_i = \frac{3}{2} \cdot 2 + 1 + 0 - 4 \overset{!}{=} 0$$

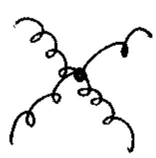
$\mathcal{L}_{3g}$



$f_i = 0 \quad , \quad b_i = 3 \, , \quad d_i = 1$

$$\longrightarrow \delta_i = 0 + 3 + 1 - 4 \overset{!}{=} 0$$

$\mathcal{L}_{4g}$



$f_i = 0 \quad , \quad b_i = 4 \, , \quad d_i = 0$

$$\longrightarrow \delta_i = 0 + 4 + 0 - 4 \overset{!}{=} 0$$

$\hookrightarrow$ QCD IS RENORMALIZABLE

⇒ RENORMALIZATION IN QCD

- $$\boxed{DD = 4 - \frac{3}{2} N_q - N_g}$$

DEGREE OF DIVERGENCE
OF ANY DIAGRAM

N_q : # EXTERNAL QUARK LINES

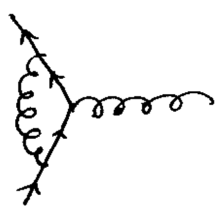
N_g : # EXTERNAL GLUON LINES

e.g.

$N_q = 2$, $N_g = 0$

$DD = 4 - \frac{3}{2} \cdot 2 - 0$

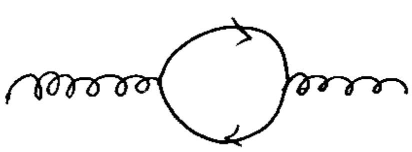
$\boxed{DD = 1}$ LINEARLY DIVERGENT



$N_q = 2$, $N_g = 1$

$DD = 4 - \frac{3}{2} \cdot 2 - 1$

$\boxed{DD = 0}$ LOGARITHMICALLY DIVERGENT



$N_q = 0$, $N_g = 2$

$DD = 4 - 0 - 2$

$\boxed{DD = 2}$ QUADRATICALLY DIVERGENT

- FOR A RENORMALIZABLE THEORY

  $\downarrow$

  ONLY A FINITE # OF DIVERGENCE TYPES APPEAR

IDEA ABSORB DIVERGENCE

BY REDEFINING A FINITE NUMBER OF PARAMETERS IN THEORY

$\hookrightarrow$ PARAMETERS (m, g) AND FIELDS (q, A^μ_a) IN $\mathcal{L}$

ARE TO BE CONSIDERED AS BARE (UNPHYSICAL) PARAMETERS

WHICH GET MODIFIED BY INTERACTION !

WRITE ORIGINAL $\mathcal{L}$ $\rightsquigarrow$ CALL $\mathcal{L}_B$ (BARE)

$$\boxed{\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{CT}}$$

$\mathcal{L}_R$: RENORMALIZED $\mathcal{L}$ PHYSICAL PARAMETERS / FIELDS

$\mathcal{L}_{CT}$: COUNTER TERM : CANCELS DIVERGENCE

$$\boxed{\begin{aligned} q_B &= Z_2^{1/2}\, q \\ A^\mu_{a,B} &= Z_3^{1/2}\, A^\mu_a \\ \chi_{a,B} &= \tilde{Z}_3^{1/2}\, \chi_a \\ g_B &= Z_g\, g \\ m_B &= Z_m\, m \\ \xi_B &= Z_3\, \xi \end{aligned}}$$

(GAUGE PARAMETER)

q_B : BARE FIELD

q : RENORMALIZED FIELD

Z_2 : FIELD RENORM. CONSTANT 'ABSORBS' DIVERGENCE

$\hookrightarrow$ e.g. $\mathcal{L}_B^{qqg} = -g_B \bar{q}_B \, \gamma^\mu \frac{\lambda_a}{2} \, q_B \, A^a_{\mu,B}$

$$= - Z_g \, Z_2 \, Z_3^{1/2} \; g \, \bar{q} \, \gamma^\mu \frac{\lambda_a}{2} \, q \, A^a_\mu$$

$$= \underbrace{- g \, \bar{q} \, \gamma^\mu \frac{\lambda_a}{2} \, q \, A^a_\mu}_{\mathcal{L}_R}$$

$$+ \underbrace{\left(Z_g Z_2 Z_3^{1/2} - 1 \right) \cdot (-g) \, \bar{q} \, \gamma^\mu \frac{\lambda_a}{2} \, q \, A^a_\mu}_{\mathcal{L}_{CT}}$$

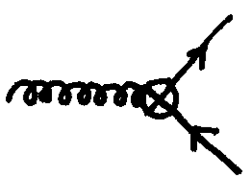
$\therefore$ INSTEAD OF CALCULATING WITH $\mathcal{L}_B$ WITH UNPHYSICAL PARAMETERS,

WE CALCULATE WITH $\mathcal{L}_R$ WITH PHYSICAL PARAMETERS

& SUPPLEMENT THIS WITH $\mathcal{L}_{CT}$ $\Rightarrow$ CHOSEN IN SUCH A WAY THAT IT ABSORBS ALL DIVERGENCES

$\Downarrow$

ADDITIONAL FEYNMAN RULES FOR $\mathcal{L}_{CT}$

μ, a 

$$\boxed{- i g \left(Z_g Z_2 Z_3^{1/2} - 1 \right) \gamma^\mu \frac{\lambda_a}{2}}$$

$\hookrightarrow$ e.g

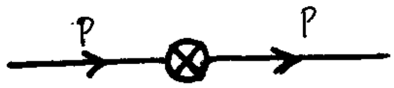
$$\mathcal{L}_{q,B} = \bar{q}_B \left(i\gamma^\mu \partial_\mu - m_B \right) q_B$$

$$= Z_2 \; \bar{q} \left(i\gamma^\mu \partial_\mu - Z_m m \right) q$$

$$= \underbrace{\bar{q} \left(i\gamma^\mu \partial_\mu - m \right) q}_{\mathcal{L}_R}$$

$$+ \underbrace{(Z_2 - 1) \; \bar{q}\, i\gamma^\mu \partial_\mu q \; - (Z_2 Z_m - 1) \; \bar{q}\, m\, q}_{\mathcal{L}_{CT}}$$

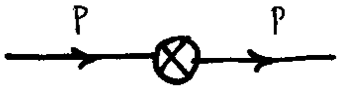
FEYNMAN RULE FOR $\mathcal{L}_{CT}$



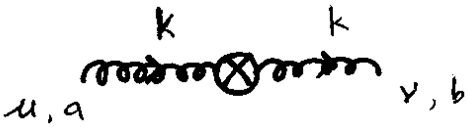
$$\boxed{i \left[(Z_2 - 1) \not{p} - (Z_2 Z_m - 1) m \right]}$$

- FEYNMAN RULES FOR COUNTERTERMS

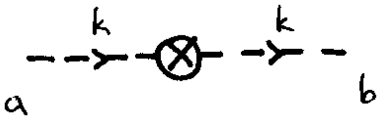
FEYNMAN RULE



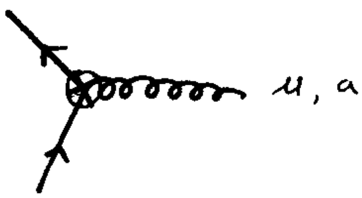
$$i\left[(Z_2-1)\not p - (Z_2 Z_m - 1)m\right]$$



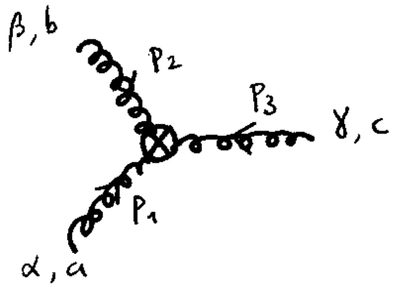
$$-i\,(Z_3-1)\,\delta_{ab}\,\left(k^2 g^{\mu\nu} - k^\mu k^\nu\right)$$



$$-i\,(\tilde Z_3 - 1)\,\delta_{ab}\,k^2$$

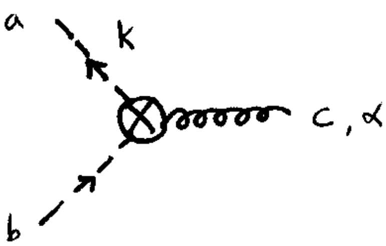


$$-ig\,(Z_{1F}-1)\,\gamma^\mu \frac{\lambda_a}{2} \qquad \text{WITH } Z_{1F} \equiv Z_g Z_2 Z_3^{1/2}$$

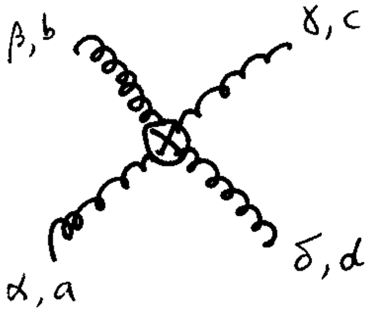


$$-g\,(Z_1-1)\,f_{abc}\,V^{\alpha\beta\gamma}(p_1,p_2,p_3)$$

$$\text{WITH } Z_1 \equiv Z_g Z_3^{3/2}$$



$$-g\,(\tilde Z_1 - 1)\,f_{abc}\,k^\alpha \qquad \text{WITH } \tilde Z_1 = Z_g \tilde Z_3 Z_3^{1/2}$$



$$-ig^2\,(Z_4-1)\,W^{\alpha\beta\gamma\delta}_{abcd} \qquad \text{WITH } Z_4 = Z_g^2 Z_3^2$$

• REGULARIZATION

CALCULATE LOOPS WITH $\mathcal{L}_R$

$\downarrow$

IN INTERMEDIATE STEP : REGULARIZE THEORY

↳ CUT-OFF METHODS $\int^{\Lambda} d^4\ell$

$\Downarrow$

VIOLATES LORENTZ-INVARIANCE

↳ PAULI-VILLARS REGULARIZATION

e.g. $$\frac{1}{\ell^2 - m^2 + i\varepsilon} \longrightarrow \frac{1}{\ell^2 - m^2 + i\varepsilon} \cdot \underbrace{\left(\frac{m^2 - \Lambda^2}{\ell^2 - \Lambda^2 + i\varepsilon} \right)}_{\substack{\downarrow \\ 1}}$$

IMPROVES CONVERGENCE BY 2 POWERS

$\Lambda \to \infty$ IN END

↳ DIMENSIONAL REGULARIZATION (T'HOOFT , VELTMAN)

CONSIDER $\int d^4\ell \longrightarrow \int d^D\ell$

CALCULATE LOOPS IN D- DIMENSIONS

$D < 4$: $\Longrightarrow$ IMPROVES CONVERGENCE

DIVERGENCE WILL APPEAR AS POLE IN $\left(\varepsilon \equiv 2 - \frac{D}{2} \right)$

" $\frac{1}{\varepsilon}$ " $\quad D < 4 \Longrightarrow \varepsilon > 0$

AT THE END : WHEN DIVERGENCE IS ABSORBED IN COUNTERTERM

$\downarrow$

ONE CAN LET $\varepsilon \to 0$ (ANALYTICAL CONTINUATION)

$\Rightarrow$ DIRAC ALGEBRA IN N - DIM.

$$\Rightarrow \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} I_N$$

$$\Rightarrow \quad \gamma_\mu \gamma^\mu = N I_N$$

$$\gamma_\mu \gamma_\alpha \gamma^\mu = (2 - N) \gamma_\alpha$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma^\mu = 4 g_{\alpha\beta} I_N + (N - 4) \gamma_\alpha \gamma_\beta$$

$$\gamma_\mu \gamma_\alpha \gamma_\beta \gamma_\lambda \gamma^\mu = -2 \gamma_\lambda \gamma_\beta \gamma_\alpha - (N - 4) \gamma_\alpha \gamma_\beta \gamma_\lambda$$

$$\Rightarrow \quad \mathrm{Tr}(I_N) = 4 + f(N)$$

↳ DOES NOT AFFECT LIMIT $N \to 4$

UV DIVERGENCES

⇒ FEYNMAN PARAMETRIZATION

$$\frac{1}{A_0 A_1 \dots A_n} = \Gamma(n+1) \int_0^1 dz_1 \int_0^{z_1} dz_2 \dots \int_0^{z_{n-1}} dz_n \frac{1}{\left[A_0 + (A_1 - A_0) z_1 + \dots + (A_n - A_{n-1}) z_n\right]^{n+1}}$$

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\left[B + (A-B)x\right]^{\alpha+\beta}}$$

⇒ INTEGRALS

* $$\int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[k^2 + s + i\varepsilon\right]^n} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2)}{\Gamma(n)} \frac{1}{s^{n-D/2}}$$

* $$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu}{\left[k^2 + s + i\varepsilon\right]^n} = 0$$

* $$\int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu}{\left[k^2 + s + i\varepsilon\right]^n} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(n - D/2 - 1)}{2\Gamma(n)} \frac{g^{\mu\nu}}{s^{n-D/2-1}}$$

$$\int \frac{d^D k}{(2\pi)^D} \frac{(k^2)^r}{\left[k^2 + s + i\varepsilon\right]^n} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(r + D/2)}{\Gamma(D/2)} \cdot \frac{\Gamma(n - D/2 - r)}{\Gamma(n)\, s^{n-r-D/2}}$$

$$\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left\{ \frac{1}{\varepsilon} + \psi(n+1) + O(\varepsilon) \right\}$$

$$\psi(n+1) = 1 + \dots + \frac{1}{n} - \gamma_E \qquad \gamma_E = 0.577$$

$$\psi(1) = -\gamma \qquad \Rightarrow \Gamma(\varepsilon) = \left\{ \frac{1}{\varepsilon} - \gamma_E \right\}$$

$$\Gamma(1 + \varepsilon) = 1 - \gamma_E \varepsilon + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right)\varepsilon^2 + O(\varepsilon^3)$$

* MEASURE IN D DIM EUCLIDEAN SPACE $\theta_1, \dots \theta_{D-2} : 0 \to \pi$

$\phi : 0 \to 2\pi$

$$d^D \bar{p} = |\bar{p}|^{D-1} \sin^{D-2}\theta_1 \dots \sin\theta_{D-2} \; d|\bar{p}| \, d\theta_1 \dots d\theta_{D-2} \, d\phi$$

$$\int_0^\pi d\theta \; \sin^n\theta = \sqrt{\pi} \; \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)}$$

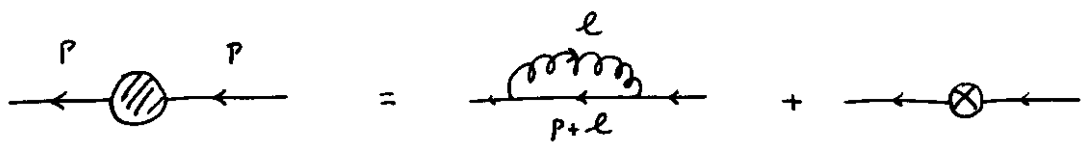
$$\delta^D(\bar{p} - \bar{p}_0) = \frac{1}{|\bar{p}|^{D-1}} \, \delta(|\bar{p}| - |\bar{p}_0|) \; \delta^{D-1}(\Omega_{\hat{p}} - \Omega_{\hat{p}_0})$$

'HYPERSURFACE' $\displaystyle \int_{(D-1)} d\Omega = \frac{2\pi^{D/2}}{\Gamma(D/2)}$

* BETA - FUNCTION

$$B(n,m) = \int_0^1 dx \; x^{n-1} (1-x)^{m-1} = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}$$

QUARK SELF-ENERGY



$$-i\Sigma_Q(p) = -i\Sigma_{Q,\ell}(p) - i\Sigma_{Q,ct}(p)$$

$$\Rightarrow -i\Sigma_{Q,\ell}(p) = \int \frac{d^D\ell}{(2\pi)^D}\, \frac{i}{\ell^2}\left[-g_{\mu\nu} + (1-\xi)\,\ell_\mu \ell_\nu/\ell^2\right] \underbrace{\left(\frac{\lambda_a}{2}\frac{\lambda_a}{2}\right)}_{\frac{4}{3}\mathbb{1}} \left[-ig\gamma^\mu\right] \frac{i(\not p + \not \ell + m)}{(p+\ell)^2 - m^2} \left[-ig\gamma^\nu\right]$$

$$= \frac{4}{3} g^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2\left[(p+\ell)^2 - m^2\right]} \left\{ -\gamma_\mu(\not p + \not \ell + m)\gamma^\mu + \frac{(1-\xi)}{\ell^2}\not \ell(\not p + \not \ell + m)\not \ell \right\}$$

$$= \frac{4}{3} g^2 \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\ell^2\left[(p+\ell)^2 - m^2\right]} \left\{ -(2-D)(\not p + \not \ell) - Dm + (1-\xi)(-\not p + m) + (1-\xi)\frac{2p\cdot\ell + \ell^2}{\ell^2}\not \ell \right\}$$

$$= \frac{4}{3} g^2 \int_0^1 dx \int \frac{d^D\ell}{(2\pi)^D} \frac{1}{\left[(\ell + px)^2 + p^2 x(1-x) - m^2 x\right]^2}$$

$$\cdot \left\{ -(2-D)(\not p + \not \ell) - Dm + (1-\xi)(-\not p + m) + (1-\xi)\not \ell\left(1 + \frac{2p\cdot\ell}{\ell^2}\right) \right\}$$

$$= \frac{4}{3} g^2 \int_0^1 dx \left\{ \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{\left[\ell^2 + p^2 x(1-x) - m^2 x\right]^2} \right.$$

$$\cdot \left\{ -(2-D)\left(\not{p}(1-x) + \cancel{\not{\ell}}\right) - D m \right.$$

$$\left. + (1-\xi)(-\not{p} + m) + (1-\xi)(\cancel{\not{\ell}} - \not{p} x) \right\}$$

$$\left. + 2(1-x) \int \frac{d^D \ell}{(2\pi)^D} \frac{(1-\xi)}{\left[\ell^2 + p^2 x(1-x) - m^2 x\right]^3} \underbrace{\left[2p\cdot(\ell - px)(\not{\ell} - \not{p}x)\right]}_{2p\cdot\ell\,\not{\ell} \;+\; 2p^2x^2\not{p} \;+\; \cdots} \right\}$$

$$= \frac{4}{3} g^2 \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \left\{ \int_0^1 dx \left[2\not{p}(1-x) - 4m + (1-\xi)(-\not{p}(1+x) + m) \right.\right.$$

$$\left.\left. + \frac{1}{2}(1-x)(2\not{p})(1-\xi) \right] \right\} + \text{FINITE TERMS}$$

$$= \frac{4}{3} g^2 \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \left\{ \not{p}\,\xi - m(3+\xi) \right\} + \text{FINITE TERMS}$$

$$\boxed{-i \Sigma_{Q,\ell}(p) = \frac{4}{3} i \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} \left\{ \not{p}\,\xi - m(3+\xi) \right\} + \text{FINITE TERMS}}$$

$$\Rightarrow \; -i \Sigma_{Q,ct}(p) = i \left[(Z_2 - 1)\not{p} - (Z_2 Z_m - 1) m \right]$$

$\Downarrow$

DIVERGENT TERM OF $\sum_{Q,\ell}$ IS CANCELLED BY COUNTERTERM

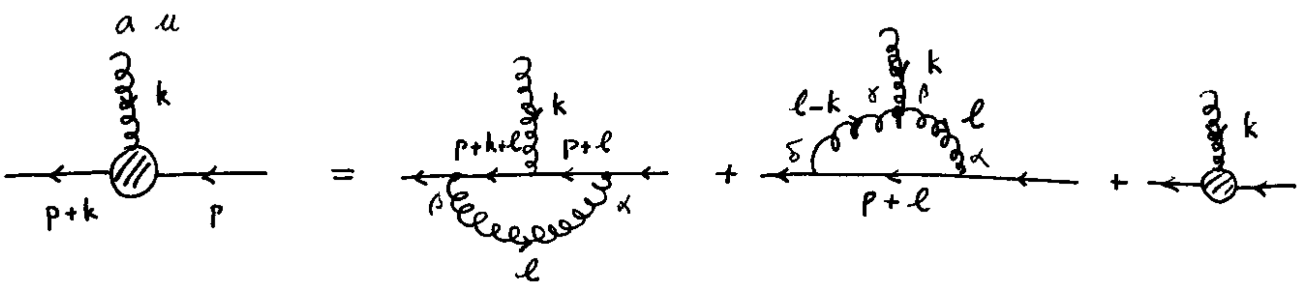
$$\boxed{Z_2 = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} \frac{4}{3} \xi + O(g^4)}$$

$$Z_2 Z_m = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} \frac{4}{3}(3 + \xi) + O(g^4)$$

$$Z_m = \left[1 - \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} \frac{4}{3}(3 + \xi) + O(g^4)\right]\left[1 + \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} \frac{4}{3}\xi + O(g^4)\right]$$

$$\boxed{Z_m = 1 - \frac{g^2}{(4\pi)^2} \frac{1}{\varepsilon} 3.\left(\frac{4}{3}\right) + O(g^4)}$$

QUARK - GLUON VERTEX



$$\Lambda^{\mu}_{a\,Q\text{-}GL}\,(p+k,p,k) = \Lambda^{\mu(1)}_{a,Q-GL} + \Lambda^{\mu(2)}_{a,Q-GL} + \Lambda^{\mu(3)}_{a,Q-GL}$$

$$\Longrightarrow \quad \Lambda^{\mu(1)}_{a,Q-GL}\,(p+k,\,p,\,k)$$

$$= \int \frac{d^D\ell}{(2\pi)^D} \left[-ig\frac{\lambda_b}{2}\gamma^\beta\right] \frac{i\left[(\not p+\not k+\not \ell)+m\right]}{(p+k+\ell)^2-m^2} \left[-ig\frac{\lambda_a}{2}\gamma^\mu\right]$$

$$\cdot \frac{i\left[\not p+\not \ell+m\right]}{(p+\ell)^2-m^2} \left[-ig\frac{\lambda_b}{2}\gamma^\alpha\right] \frac{i}{\ell^2}\left[-g_{\alpha\beta}+(1-\xi)\frac{\ell_\alpha\ell_\beta}{\ell^2}\right]$$

$$= g^3\,\frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{\gamma^\beta\left[\not p+\not k+\not \ell+m\right]\gamma^\mu\left[\not p+\not \ell+m\right]\gamma^\alpha}{\left[(p+k+\ell)^2-m^2\right]\left[(p+\ell)^2-m^2\right]\ell^2} \left[-g_{\alpha\beta}+(1-\xi)\frac{\ell_\alpha\ell_\beta}{\ell^2}\right]$$

$\Downarrow$

TO CALCULATE ONLY DIVERGENT PART

$$= g^3 \frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2} \longrightarrow -\frac{1}{6}\left(\frac{\lambda_a}{2}\right)$$

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{-\gamma^\alpha \not{\ell}\, \gamma^\mu \not{\ell}\, \gamma_\alpha + (1-\xi)\,\gamma^\mu \ell^2}{\left[(p+k+\ell)^2 - m^2\right]\left[(p+\ell)^2 - m^2\right]\ell^2} \quad + \text{ FINITE TERMS}$$

$$= -\frac{1}{6}\left(\frac{\lambda_a}{2}\right) g^3 \int \frac{d^D\ell}{(2\pi)^D} \frac{2\not{\ell}\,\gamma^\mu \not{\ell} + (1-\xi)\,\ell^2\gamma^\mu}{\left[(p+k+\ell)^2 - m^2\right]\left[(p+\ell)^2 - m^2\right]\ell^2}$$

$$+ \text{ FINITE TERMS}$$

$$= -\frac{1}{6}\left(\frac{\lambda_a}{2}\right) g^3 \; 2\int_0^1 dx\, x \int_0^1 dy \int \frac{d^D\ell}{(2\pi)^D} \frac{2\not{\ell}\,\gamma^\mu \not{\ell} + (1-\xi)\,\ell^2\gamma^\mu}{\left[\ell^2 + \dots\right]^3} + \text{ FINITE TERMS}$$

$$= -\frac{1}{6}\left(\frac{\lambda_a}{2}\right) g^3 \frac{i}{(4\pi)^2}\; \frac{1}{\varepsilon}\; \frac{1}{4} \left\{ + 2\,\gamma^\alpha\gamma^\mu\gamma_\alpha + (1-\xi)\,4\,\gamma^\mu \right\}$$

$$+ \text{ FINITE TERMS}$$

$$\boxed{\Lambda^{\mu\,(1)}_{a,\,Q\text{-}GL}(p+k, p, k) = -ig\,\frac{\lambda_a}{2}\,\gamma^\mu \left[\frac{1}{\varepsilon}\,\frac{g^2}{(4\pi)^2}\left(-\frac{\xi}{6}\right)\right] + \text{ FINITE TERMS}}$$

$$\frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2}$$

$$= i f_{bac}\frac{\lambda_c}{2}\frac{\lambda_b}{2} + \frac{\lambda_a}{2}\underbrace{\frac{\lambda_b}{2}\frac{\lambda_b}{2}}_{\frac{4}{3}\mathbb{1}}$$

$$\Rightarrow f_{bac}\frac{\lambda_c}{2}\frac{\lambda_b}{2} = f_{bac}\frac{1}{2}\left(\frac{1}{3}\delta_{bc} + (d_{cbd} + i f_{cbd})\frac{\lambda_d}{2}\right)$$

$$= \frac{i}{2}\underbrace{f_{bac}f_{cbd}}_{3\delta_{ad}}\frac{\lambda_d}{2}$$

$$= \frac{3}{2}i\frac{\lambda_a}{2}$$

$$\therefore \quad \frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2} = -\frac{3}{2}\frac{\lambda_a}{2} + \frac{4}{3}\frac{\lambda_a}{2}$$

$$\frac{\lambda_b}{2}\frac{\lambda_a}{2}\frac{\lambda_b}{2} = -\frac{1}{6}\frac{\lambda_a}{2}$$

$$\Rightarrow \Lambda^{\mu(2)}_{a,\,Q-GL}(p+k,p,k)$$

$$= \int \frac{d^D\ell}{(2\pi)^D} \frac{i}{(\ell-k)^2} \left[-g_{\gamma\delta} + (1-\xi)(\ell-k)_\gamma (\ell-k)_\delta / (\ell-k)^2\right]$$

$$\cdot \frac{i}{\ell^2} \left[-g_{\beta\alpha} + (1-\xi)\, \ell_\alpha \ell_\beta / \ell^2\right]$$

$$\cdot \left[-g f_{bac} V^{\gamma\mu\beta}(\ell-k, k, -\ell)\right]$$

$$\cdot \left[-ig\gamma^\delta \frac{\lambda_b}{2}\right] i \frac{(\not p + \not \ell) + m}{(p+\ell)^2 - m^2} \left[-ig\gamma^\alpha \frac{\lambda_c}{2}\right]$$

$$= + i g^3 f_{abc} \frac{\lambda_b}{2}\frac{\lambda_c}{2}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell-k)^2 \ell^2 \left[(p+\ell)^2 - m^2\right]}$$

$$\cdot \left[-g_{\gamma\delta} + (1-\xi)(\ell-k)_\gamma (\ell-k)_\delta / (\ell-k)^2\right]$$

$$\cdot \left[-g_{\alpha\beta} + (1-\xi)\, \ell_\alpha \ell_\beta / \ell^2\right] V^{\gamma\mu\beta}(\ell-k, k, -\ell)$$

$$\cdot \gamma^\delta \left[(\not p + \not \ell) + m\right] \gamma^\alpha$$

$$\Lambda^{\mu(2)}_{a,\,Q-GL}\,(p+k,\,p,\,k)$$

$$= i g^3 f_{abc} \frac{\lambda_b}{2} \frac{\lambda_c}{2}$$

$$\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell-k)^2 \ell^2 \left[(p+\ell)^2 - m^2\right]}$$

$$\cdot \Big\{ 2\gamma^\mu \ell^2 + 4 \ell^\mu \not{\ell}$$

$$- \frac{(1-\xi)}{(\ell-k)^2} \ell^2 \left[\ell^2 \gamma^\mu - \ell^\mu \not{\ell}\right]$$

$$- \frac{(1-\xi)}{\ell^2} \ell^2 \left[\ell^2 \gamma^\mu - \ell^\mu \not{\ell}\right] \Big\} + \text{FINITE TERMS}$$

$$= i g^3 f_{abc} \frac{\lambda_b}{2} \frac{\lambda_c}{2}$$

$$\cdot \Big\{ 2 \int_0^1 dx\, x \int_0^1 dy \frac{i}{(4\pi)^2} \frac{1}{4} \frac{1}{\varepsilon} \left[2\gamma^\mu \cdot 4 + 4\gamma^\mu\right]$$

$$- (1-\xi)\, 2 \int_0^1 dx\, (1-x) \frac{i}{(4\pi)^2} \frac{1}{4} \frac{1}{\varepsilon} \left[4\gamma^\mu - \gamma^\mu\right] \times 2 \Big\}$$

$+$ FINITE TERMS

$$\frac{i}{2} f_{bcd}\frac{\lambda_d}{2} + \frac{1}{2} d_{bcd}\frac{\lambda_d}{2} + \frac{1}{6}\delta_{bc}$$

$$= i g^3 \left(f_{abc} \left(\frac{\lambda_b}{2}\frac{\lambda_c}{2}\right) = i\,\frac{3}{2}\frac{\lambda_a}{2} \right)$$

$$\cdot \frac{i}{(4\pi)^2}\,\frac{1}{\varepsilon} \left\{ \underbrace{3\gamma^\mu - (1-\xi)\,\frac{3}{2}\,\gamma^\mu}_{\frac{3}{2}\gamma^\mu (1+\xi)} \right\} + \text{FINITE TERMS}$$

$$\boxed{\Lambda^{\mu(2)}_{a,Q-GL}(p+k,p,k) = -ig\,\frac{\lambda_a}{2}\,\gamma^\mu \left[\frac{1}{\varepsilon}\,\frac{g^2}{(4\pi)^2}\,\frac{9}{4}(1+\xi)\right] + \text{FINITE TERMS}}$$

$$\Rightarrow \boxed{\Lambda^{\mu(3)}_{a,Q-GL}(p+k,p,k) = -ig\,\frac{\lambda_a}{2}\,\gamma^\mu \left(Z_{1F} - 1\right)}$$

$$\therefore \boxed{Z_{1F} = 1 - \frac{g^2}{(4\pi)^2}\,\frac{1}{\varepsilon}\,\underbrace{\left[\frac{9}{4} + \frac{25}{12}\,\xi\right]}_{\text{FOR } \xi = 1:\ \frac{13}{3}} + O(g^4)}$$