

1. Harmonic oscillator in terms of the path integral

Consider the one-dimensional harmonic oscillator:

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2.$$

- (a) {4pts} Compute $K(q_f, q_i; T)$ by the discretized version of the Feynman's path-integral formula (see Eq. (1-45) on page 10 of the lecture notes).
- (b) {3pts} Solve the Euler-Lagrange equation to find the classical trajectory $q^{(class.)}(t)$, with the boundary conditions $q(0) = q_i$, $q(T) = q_f$. Use the Dirac's formula to find $K(q_f, q_i; T)$, with the prefactor determined from

$$\int_0^\infty dq K(q_f, q; T/2) K(q, q_i; T/2) = K(q_f, q_i; T).$$

2. Cross section of elastic scattering in ϕ^4 theory

Consider the elastic scattering of 2 particles in ϕ^4 theory in the center-of-mass frame. In this case the differential cross section is defined as

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}|^2, \quad (1)$$

where \mathcal{M} is the scattering amplitude, s is the Mandelstam variable. Thus the total cross section is

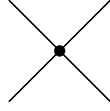
$$\sigma = \frac{1}{2} \int d\Omega \frac{d\sigma}{d\Omega}. \quad (2)$$

The factor 1/2 arises here because of the 2 identical particles in the final state (see Peskin & Schroeder, last paragraph of section 4.5 and Eqs. (4.99), (4.100) for details).

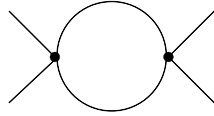
- (a) {1pt} The tree-level amplitude is given by Fig.1, hence

$$\mathcal{M} = -\lambda.$$

Calculate the total cross section for this case.

Figure 1: Tree level diagram in ϕ^4 -theory

- (b) {2pts} Find the symmetry factor of the diagram in Fig.2.

Figure 2: One-loop diagram in ϕ^4 theory

- (c) {3pts} Calculate the diagram in Fig.2 using dimensional regularization. Use the Feynman-parameter trick:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(xA + (1-x)B)^2}.$$

Compare the results numerically with *LoopTools* integral B_0 .

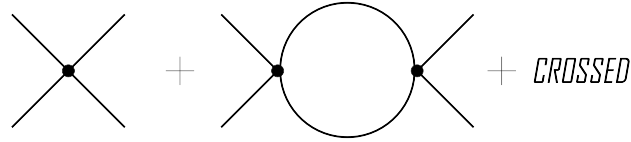
- (d) {3pts} Determine the imaginary part of this one loop diagram. When and why does it emerge? (Hint: unitarity)

Obtain the imaginary part of Fig.2 using the optical theorem, i.e, the imaginary part of the forward scattering amplitude is given by the total cross section:

$$\text{Im } \mathcal{M}^{\text{one-loop}} = 2s |\vec{p}_{cm}| \sigma^{\text{tree}}(s)$$

where $|\vec{p}_{cm}| = \frac{1}{2}\sqrt{s - 4m^2}$ is the center-of-mass momentum. The result for the cross section is from part (a) of the problem.

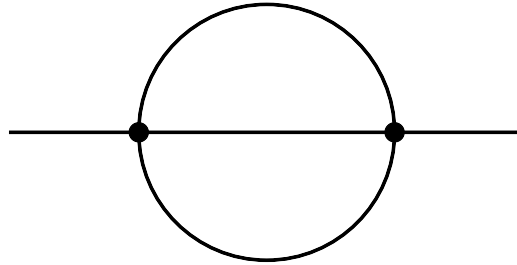
- (e) {4pts} Compute the cross section of two-particle scattering using
- LoopTools*
- in the next-to-leading order (Fig.3). Renormalize the result, such that the physical value of
- λ
- is given by the scattering length (i.e., the 0-energy scattering amplitude). This means the entire amplitude at threshold (
- $s = 4m^2$
-) is given by
- λ
- .

Figure 3: NLO two particle scattering in ϕ^4 -theory

3. Two-loop self-energy calculation in the ϕ^4 theory

The “sunset diagram” contribution to the self-energy in ϕ^4 theory (Fig.4) reads as

$$\Sigma(p^2) = -\frac{\lambda^2}{6} \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - m^2) [(k - \ell)^2 - m^2] [(p - k)^2 - m^2]}$$

Figure 4: Two-loop “sunset” diagram of ϕ^4 theory

- (a) {1pt} Justify the symmetry factor of this diagram: 6.
- (b) {5pts} Calculate the “sunset diagram” contribution in the **massless** case ($m^2 = 0$) using dimensional regularization. The following form of the Feynman parametrization will be helpful after the first integration over the 4-momentum,

$$\frac{1}{AB^\alpha} = \alpha \int_0^1 \frac{x^{\alpha-1} dx}{(xB + (1-x)A)^{1+\alpha}}$$

Show that the result can be written as:

$$\begin{aligned} \Sigma(p^2) &= -\frac{\lambda^2 p^2}{6(4\pi)^4} \left(-\frac{4\pi\mu^2}{p^2} \right)^{2\epsilon} \frac{\Gamma(-1+2\epsilon)\Gamma^3(1-\epsilon)}{\Gamma(3-3\epsilon)} \\ &= -\frac{\lambda^2 p^2}{6(4\pi)^4} \frac{1}{2} \left[-\frac{1}{2\epsilon} + \gamma_E - \frac{13}{4} + \log \left(-\frac{p^2}{4\pi\mu^2} \right) + O(\epsilon) \right] \end{aligned}$$

Guide for Windows users how to install the *LoopTools* package in Mathematica

1. Download the file **LoopTools-Cygwin.exe.gz** from the official site <http://www.feynarts.de/looptools/>.
2. Unpack the archive by any appropriate program (e.g. WinRAR) to get the executable **LoopTools-Cygwin.exe**.
3. Find the **cygwin1.dll** library in the Internet and download it.
4. Put the **cygwin1.dll** and **LoopTools-Cygwin.exe** in the same directory.
5. Open Mathematica notebook, and type
Install["full path to the directory with LoopTools//LoopTools.exe"].
Be aware that Mathematica has its own rules for slashes to define the path!
6. Run this command by Shift+Enter (as usual for all commands in Mathematica).
7. Choose the file **LoopTools-Cygwin.exe** in the opened window.

ALSO! Anyway you should install Cygwin for the further problems that will require FORM! So read something about it in the Internet and try to install it by yourself. The main problem is to install all important compilers and libraries in Cygwin (e.g. gcc and gfortran) to give an ability to FORM to compile its internal routines.