Cosmology and General Relativity: HW 6 (bonus)

Solutions

Problem 1 Internal Schwarzschild solution: The Schwarzschild solution is only valid outside of a spherically symmetric mass distribution. In this exercise we seek the continuation into the interior of a massive object (e.g. a star). We assume the object to be a perfect fluid, i.e. the energy momentum tensor is given by:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \ .$$

For the metric, we use the ansatz

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{1}$$

with $A(r) = e^{\nu(r)}$ and $B(r) = e^{\lambda(r)}$.

a. Show that matter at rest in these coordinates has the following four-velocity:

$$(u^\mu) = (-e^{-\nu/2}, 0, 0, 0) \ .$$

1pt

Inserting the components of the Ricci tensor into Einstein's equations gives the following set of differential equations ($\kappa \equiv 8\pi G$):

$$-\kappa\rho = -e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] - \frac{1}{r^2} \ , \tag{2}$$

$$\kappa p = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} \ , \tag{3}$$

$$\kappa p = e^{-\lambda}\left[\frac{\nu''}{2} + \frac{(\nu')^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu' - \lambda'}{2r}\right] \ . \tag{4}$$

b. Show that from energy momentum conservation ($\nabla_\mu T^{\mu\nu} = 0$), if follows:

$$p' = -\frac{\nu'}{2}(p + \rho) \ . \tag{5}$$

Since this equation follows from the field equations (2), (3), and (4), you can continue with this equation instead of one of the field equations. *2pt*

c. Show that the solution to (2) is given by:

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r} + \frac{C}{r} \qquad \text{with} \qquad m(r) = \frac{\kappa}{2}\int_0^r \rho(r')r'^2 dr' \ ,$$

and some integration constant C. Observe that $m(r)$ is the mass contained within the radius r (up to some factors of G and c). The requirement that g^{rr} is finite at $r = 0$ leads to $C = 0$. *3pt*

For the fluid, an equation of state $f(\rho, p) = 0$ has to be chosen. For simplicity, we assume a constant mass density,

$$\rho = \text{const.} \tag{6}$$

Note that this equation of state does not provide a realistic model. A constant mass density is a first approximation for small stars, in which the pressure is not too high. The spherically symmetric static solution to the equation of state (6) is the *interior Schwarzschild solution*.
With the help of the equation of state (6), the solution to (2) can be simplified to:

$$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r} \qquad \text{with} \qquad m(r) = \frac{\kappa\rho r^3}{6} \ , \tag{7}$$

and (5) can be integrated to:

$$p + \rho = Be^{-\nu/2} \qquad \text{with integration constant } B \ . \tag{8}$$

A linear combination of the equations (2), (3), and (4) can now be taken as the third independent differential equation.

d. Show that from linear combination $-(2) + (3)$, it follows:

$$\left[e^{\nu/2}\left(1 - \frac{2m}{r}\right)^{-1/2}\right]' = \frac{\kappa B r}{2(1 - \frac{2m}{r})^{3/2}} \ .$$

Derive the solution

$$e^{\nu/2} = \frac{r^3}{4m}\kappa B - D\sqrt{1 - \frac{2m}{r}} \tag{9}$$

where D is another integration constant. *4pt*

Solution:

a. Starting with the metric (with the spatial part omitted for matter at rest):

$$\begin{aligned} d\tau^2 &= A(r)dt^2 \\ d\tau &= \pm e^{\nu(r)/2}dt \\ \pm e^{-\nu(r)/2} &= \frac{dt}{d\tau} \ . \end{aligned}$$

Choosing the four-velocity in the forward-time direction:

$$\boxed{(u^\mu) = (-e^{-\nu(r)/2}, 0, 0, 0)} \ .$$

b. First, since $u_\mu u^\mu = 1$, then $u_\mu = (-e^{\nu(r)/2}, 0, 0, 0)$. Writing out the stress-energy tensor (non-zero components):

$$\begin{aligned} T_{tt} &= (\rho + p)e^\nu - pe^\nu = \rho e^\nu \\ T_{ii} &= -pg_{ii} \ . \end{aligned}$$

Raising the indices:

$$\begin{aligned} T^{\mu\nu} &= T_{\rho\sigma} g^{\rho\mu} g^{\sigma\nu} \\ &= T_{tt} g^{t\mu} g^{t\nu} + T_{ii} g^{i\mu} g^{i\nu} \\ &= \delta^\mu_t \delta^\nu_t e^{-\nu} \rho + \delta^\mu_i \delta^\nu_i g^{ii} p \ . \end{aligned}$$

From energy-momentum conservation

$$\begin{aligned} 0 &= \nabla_\mu T^{\mu\nu} \\ &= \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\rho} T^{\rho\nu} + \Gamma^\nu_{\mu\rho} T^{\mu\rho} \\ &= \partial_t T^{t\nu} + \Gamma^\nu_{tt} T^{tt} + \Gamma^\mu_{\mu t} T^{t\nu} + \partial_i T^{i\nu} + \Gamma^\mu_{\mu i} T^{i\nu} + \Gamma^\nu_{ii} T^{ii} \ . \end{aligned}$$

From the previous homework,

$$\begin{aligned} \Gamma^t_{\mu\nu} &= \frac{A'}{2A}(\delta^r_\mu \delta^t_\nu + \delta^t_\mu \delta^r_\nu) \\ \Gamma^r_{\mu\nu} &= \frac{1}{2B}\left(A'\delta^t_\mu \delta^t_\nu + B'\delta^r_\mu \delta^r_\nu - 2r(\delta^\vartheta_\mu \delta^\vartheta_\nu + \delta^\varphi_\mu \delta^\varphi_\nu \sin^2\vartheta)\right) \\ \Gamma^\vartheta_{\mu\nu} &= \frac{1}{r}(\delta^r_\mu \delta^\vartheta_\nu + \delta^r_\nu \delta^\vartheta_\mu) - \sin\vartheta\cos\vartheta \ \delta^\varphi_\mu \delta^\varphi_\nu \\ \Gamma^\varphi_{\mu\nu} &= \frac{1}{r}(\delta^r_\mu \delta^\varphi_\nu + \delta^r_\nu \delta^\varphi_\mu) + (\delta^\vartheta_\mu \delta^\varphi_\nu + \delta^\vartheta_\nu \delta^\varphi_\mu)\cot\vartheta \ . \end{aligned}$$

Now, $\nu = t$ is trivial. For $\nu = j$:

$$\begin{aligned} 0 &= \Gamma^j_{tt} T^{tt} + \partial_j T^{jj} + \Gamma^\mu_{\mu j} T^{jj} + \Gamma^j_{ii} T^{ii} \\ &= \Gamma^r_{tt} T^{tt} \delta^j_r + \partial_r T^{rr} \delta^j_r + \Gamma^\mu_{\mu r} T^{rr} \delta^j_r + \Gamma^\varphi_{\varphi\vartheta} T^{\vartheta\vartheta} \delta^j_\vartheta + \Gamma^r_{ii} T^{ii} \delta^j_r + \Gamma^\vartheta_{\varphi\varphi} T^{\varphi\varphi} \delta^j_\vartheta \ . \end{aligned}$$

Clearly $j = \varphi$ is a trivial equation. For $j = \vartheta$:

$$\begin{aligned} 0 &= \Gamma^\varphi_{\varphi\vartheta} T^{\vartheta\vartheta} + \Gamma^\vartheta_{\varphi\varphi} T^{\varphi\varphi} \\ &= pr^{-2}\left(\cot\vartheta - \frac{\sin\vartheta\cos\vartheta}{\sin^2\vartheta}\right) \ , \end{aligned}$$

which is trivial. Finally, there is the case with $j = r$. For this:

$$\begin{aligned}
\partial_r T^{rr} &= -\partial_r \left(pe^{-\lambda}\right) \\
&= e^{-\lambda}\left(p\lambda' - p'\right) \\
\Gamma^{\mu}_{\mu r} T^{rr} &= -pe^{-\lambda(r)}\left(\frac{\nu'(r)+\lambda'(r)}{2} + \frac{2}{r}\right) \\
\Gamma^{r}_{\mu\mu} T^{\mu\mu} &= -\frac{1}{2}\nu' e^{\nu-\lambda} e^{-\nu}\rho - \frac{1}{2}\lambda' e^{-\lambda} p + pre^{-\lambda}\left(\frac{1}{r^2} + \frac{\sin^2\vartheta}{r^2\sin^2\vartheta}\right) \\
&= e^{-\lambda}\left(-\frac{1}{2}\nu'\rho - \frac{1}{2}\lambda' p + 2\frac{p}{r}\right) \ .
\end{aligned}$$

Adding these together:

$$\begin{aligned}
0 &= e^{-\lambda}\left[p\lambda' - p' - p\frac{\nu'+\lambda'}{2} - p\frac{2}{r} - \frac{1}{2}\nu'\rho - \frac{1}{2}\lambda' p + 2\frac{p}{r}\right] \\
0 &= -p' - p\frac{\nu'}{2} - \frac{1}{2}\nu'\rho \ .
\end{aligned}$$

Equivalently:

$$\boxed{p'(r) = -\frac{\nu'(r)}{2}(p+\rho)} \ .$$

c. From Eq. (2):

$$\begin{aligned}
-\kappa\rho &= -e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] - \frac{1}{r^2} \\
1 - \kappa\rho r^2 &= -e^{-\lambda}\left[r\lambda' - 1\right] \\
1 - \kappa\rho r^2 &= \frac{d}{dr}\left(re^{-\lambda}\right) \\
\int_0^r dr'\left(1 - \kappa\rho r'^2\right) &= re^{-\lambda} - r'e^{-\lambda(r')}\Big|_{r'=0} \\
r - \int_0^r dr'\kappa\rho r'^2 &= re^{-\lambda} - r'e^{-\lambda(r')}\Big|_{r'=0} \ .
\end{aligned}$$

Then for

$$m(r) = \frac{\kappa}{2}\int_0^r dr'\rho(r)r'^2 \quad \text{and} \quad C = \lim_{r\to 0} re^{-\lambda(r)} \ ,$$

the solution is:

$$\boxed{e^{-\lambda} = 1 - \frac{2m(r)}{r} + \frac{C}{r}} \ .$$

Observe that a finite $g^{rr} = e^{-\lambda(r)}$ at $r = 0$ means that $C = 0$. Also observe that, since $\kappa = 8\pi G_N$, m is the mass of the object within r from the origin (with $G_N = 1$ units).

d. If $\rho(r) = \rho$ is constant, $m(r) = \frac{\kappa\rho r^3}{6}$. Also, from Eq. (5),

$$\begin{aligned}\left(p'(r) + \frac{\nu'(r)}{2}(p(r)+\rho)\right)e^{-\nu(r)/2} &= 0\\ \frac{d}{dr}\left((p(r)+\rho)e^{\nu(r)/2}\right) &= 0\\ (p(r)+\rho)e^{\nu(r)/2} &= (p(r_0)) + \rho)e^{\nu(r_0)/2}\\ (p(r)+\rho) &= Be^{-\nu(r)/2}\ .\end{aligned}$$

for constant

$$B = (p(r_0)) + \rho)e^{\nu(r_0)/2}\ .$$

Now, taking $-(2) + (3)$:

$$\begin{aligned}\kappa(\rho + p) &= e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2} + \frac{\nu'}{r} + \frac{1}{r^2}\right]\\ \frac{\kappa B}{1-\frac{2m}{r}}e^{-\nu/2} &= \left[\frac{\lambda'+\nu'}{r}\right]\\ \frac{\kappa B r}{2\left(1-\frac{2m(r)}{r}\right)^{3/2}} &= \frac{1}{2}e^{\nu/2}(\lambda'+\nu')\left(1-\frac{2m(r)}{r}\right)^{-1/2}\\ &= \frac{1}{2}e^{(\nu+\lambda)/2}(\lambda'+\nu')\\ &= \frac{d}{dr}e^{(\nu+\lambda)/2}\ ,\end{aligned}$$

or

$$\boxed{\frac{d}{dr}\left[e^{\nu/2}\left(1-\frac{2m}{r}\right)^{-1/2}\right] = \frac{\kappa B r}{2\left(1-\frac{2m(r)}{r}\right)^{3/2}}}\ .$$

Integrating this with respect to r (the substitution $u = r^2$ will be useful):

$$\begin{aligned}
e^{\nu/2}\left(1-\frac{2m}{r}\right)^{-1/2} - D_0 &= \int dr \frac{\kappa B r}{2\left(1-\frac{2m(r)}{r}\right)^{3/2}} \\
&= \int dr \frac{\kappa B r}{2\left(1-\frac{\kappa\rho r^2}{3}\right)^{3/2}} \\
&= \int du \frac{\kappa B}{4\left(1-\frac{\kappa\rho u}{3}\right)^{3/2}} \\
&= \frac{\kappa B}{4}\frac{6}{\kappa\rho}\left(1-\frac{\kappa\rho u}{3}\right)^{-1/2} - D_1 \\
&= \frac{\kappa B r^3}{4m}\left(1-\frac{2m}{r}\right)^{-1/2} - D_1 \ ,
\end{aligned}$$

or

$$\boxed{e^{\nu/2} = \frac{\kappa B r^3}{4m} - D\sqrt{1-\frac{2m}{r}}} \ .$$